
Global properties of Dirichlet forms on discrete spaces

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Abstract

The goal of this Diploma thesis is to study global properties of Dirichlet forms associated with infinite weighted graphs. These include recurrence and transience, stochastic completeness and the question whether the Neumann form on a graph is regular. We show that recurrence of the regular Dirichlet form of a graph is equivalent to recurrence of a certain random walk on it. After that, we prove some general characterizations of the mentioned global properties which allow us to investigate their connections. It turns out that recurrence always implies stochastic completeness and the regularity of the Neumann form. In the case where the underlying ℓ^2 -space has finite measure, we are able to show that all concepts coincide. Finally, we demonstrate that the above properties are all equivalent to uniqueness of solutions to the eigenvalue problem for the (unbounded) graph Laplacian when considered on the right space.

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Chapter 1

Introduction

The theory of Dirichlet forms on discrete spaces emerges from the study of certain partial differential equations on manifolds (e.g. the heat equation and the eigenvalue equation for the Laplacian). Since these can almost never be solved explicitly, it is convenient to discretise them. This leads to difference equations which are described by the so called graph Laplace operator. Then properties of this operator can be deduced by investigating Dirichlet forms associated with graphs on ℓ^2 -spaces.

In this Diploma thesis we draw our attention to global properties of such forms. Its original goal was to investigate the theory of recurrence and transience of these objects and to compare several notions of recurrence found in the literature. However, it turned out, that these things are intimately related to stochastic completeness and to the question whether the maximal Dirichlet form associated with a graph is regular. Therefore, we study all these properties and their connections.

Recurrence and stochastic completeness of a Dirichlet form associated with a graph have a probabilistic motivation. They are related to the long time behaviour of a certain Markov process in continuous time corresponding to the form. Stochastic completeness describes the fact that this process does not leave the graph in finite time while recurrence says that the process spends an infinite time in every vertex. The question whether the maximal Dirichlet form is regular arises from an analogue problem in the smooth case, namely whether the Laplacian with Neumann boundary conditions coincides with the one with Dirichlet boundary conditions.

We show that the notion of recurrence of the regular Dirichlet form associated with a graph and the one of a recurrent graph, which can be found in the literature (see e.g. [Soa1] and [Woe]), coincide. In the light of that, characterizations of recurrence of a graph

which deal with properties of 'the' graph Laplace operator must hold for our notion of recurrence as well. We prove them with the help of the theory of recurrence and transience as it is presented in [FOT]. Afterwards we use these 'classical' results to provide a new characterization for recurrence, which is an analogue to a theorem of [GM] on a manifold, and to extend a theorem of [JP1], dealing with a vanishing boundary term, to the case of locally infinite graphs. It is known that stochastic completeness is equivalent to uniqueness of solutions to the eigenvalue equation for the graph Laplacian on ℓ^∞ (see [KL], [Woj]) and that regularity of the maximal Dirichlet form of a graph is related to uniqueness of such solutions on the maximal form domain (see [HKLW]). As a last result dealing with recurrence, we prove that it is equivalent to uniquely solving this equation as well. Here the space of consideration are all functions of finite energy. A main tool for showing this statement will be the fact that all the global properties discussed so far coincide in the case of finite measure.

We will proceed as follows. After introducing all the objects of interest (see Chapter 2) a first step for our research is to adapt the existing theory of recurrence and transience of Dirichlet forms in [FOT] to the discrete setting. Since the underlying ℓ^2 -space is a space of sequences we can simplify many details (see Chapter 3). We then show that the notion of a recurrent Dirichlet form on a graph and the notion of a recurrent graph (as it is commonly used in the literature) coincide in the case where the measure m is constant (see Chapter 4). There are several known characterizations of recurrent graphs. We reprove them with the help of Dirichlet form methods which even allow us to provide some new results (see Chapter 5). This is followed by a discussion of stochastic completeness and the question whether the Neumann form associated with a graph is regular (see Chapter 6). Here, we present a new proof for a result of [HKLW] as well as a new characterization. Finally, an investigation of the relation of all the global properties introduced so far concludes this Diploma thesis. We prove that all of the concepts coincide whenever m is finite and that all are related to uniquely solving the eigenvalue equation for the graph Laplace operator on the right space (see Chapter 7). For recurrence these observations seem to be new.

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Chapter 2

Forms and spaces associated with graphs

In this chapter we introduce the objects of our studies. Following [HKLW], we specify what we will call a weighted graph (b, c) over a vertex set V , define the ensemble $(\tilde{D}, \tilde{F}, \tilde{L}, \tilde{Q})$ of associated objects and show their basic connections. In the next section, we consider a certain inner product on \tilde{D} , which turns it into a Hilbert space. Since spaces of this sort were introduced in [Yam1], we will call it the Yamasaki space \mathbf{D} . We prove some results about the structure of \mathbf{D} . Finally, we introduce the notion of a Dirichlet form associated with (b, c) . Most results of this chapter are well known. However, we include their proofs for the convenience of the reader.

2.1 Basic definitions

Assume V is an infinite, countable set. Let $b : V \times V \rightarrow [0, \infty)$ be such that

- (b0) $b(x, x) = 0$ for all $x \in V$
- (b1) $b(x, y) = b(y, x)$ for all $x, y \in V$
- (b2) $\sum_{y \in V} b(x, y) < \infty$ for all $x \in V$

and let

$$c : V \rightarrow [0, \infty).$$

We call the pair (b, c) a weighted graph over the vertex set V . We say $x, y \in V$ are connected by an edge whenever $b(x, y) > 0$. In this case we write $x \sim y$ and call $b(x, y)$ the weight of the edge connecting x and y . Vertices $x \in V$ with $c(x) > 0$ might be thought of being

connected with a point ∞ which is not contained in V . We will call a finite sequence of vertices $x_0, \dots, x_n \in V$ a *path* connecting x_0 and x_n if $x_j \sim x_{j+1}$ for $j = 0, \dots, n-1$. A subset $W \subseteq V$ is said to be *connected* if for every $x, y \in W$ there is a path in W connecting x and y . Furthermore

$$\deg(x) = \sum_{y \in V} b(x, y) + c(x)$$

is said to be the *generalized vertex degree* of x . For a graph $(b, 0)$, where b might only take the values 0 or 1, $\deg(x)$ coincides with the number of edges emerging from x . A graph (b, c) over V is called *locally finite* if the sets

$$\{y \in V \mid b(x, y) > 0\}$$

are finite for every $x \in V$, i.e. each vertex is only connected with finitely many other vertices.

Let $C(V)$ be the set of all real valued functions on V . To a graph (b, c) over V we associate the quadratic form

$$\tilde{Q} := \tilde{Q}_{b,c} : C(V) \rightarrow [0, \infty]$$

given by

$$\tilde{Q}(u) = \frac{1}{2} \sum_{x,y \in V} b(x, y)(u(x) - u(y))^2 + \sum_{x \in V} u(x)^2 c(x).$$

The next lemma is crucial for further discussions. It shows that \tilde{Q} satisfies certain cut-off properties.

Lemma 2.1. *Let $C : \mathbb{R} \rightarrow \mathbb{R}$ be a normal contraction (i.e. $C(0) = 0$ and $|C(x) - C(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$). Then for $u \in C(V)$ the inequality*

$$\tilde{Q}(C \circ u) \leq \tilde{Q}(u)$$

holds.

Proof. A direct calculation yields the statement. □

Remark 2.2. • Typical examples for such normal contractions are $C(x) = |x|$ and $C(x) = (0 \wedge x) \vee 1$ (here $a \wedge b = \max\{a, b\}$ and $a \vee b = \min\{a, b\}$).

- The above lemma is important for showing that certain ℓ^2 -restrictions of \tilde{Q} are Dirichlet forms.

We will be interested in the space of all *functions of finite energy*, which is defined as

$$\tilde{D} = \{u \in C(V) \mid \tilde{Q}(u) < \infty\}.$$

For $x \in V$ let δ_x be the function on V , which vanishes everywhere except in x , where it takes value 1. Then obviously

$$\tilde{Q}(\delta_x) = \deg(x).$$

Therefore the assumption (b2) implies that $C_c(V)$, the finitely supported functions, are contained in \tilde{D} . Abusing notation, we can extend \tilde{Q} to a bilinear map

$$\tilde{Q} : \tilde{D} \times \tilde{D} \rightarrow \mathbb{R}$$

acting as

$$\tilde{Q}(u, v) = \frac{1}{2} \sum_{x, y \in V} b(x, y)(u(x) - u(y))(v(x) - v(y)) + \sum_{x \in V} u(x)v(x)c(x).$$

The above sum is absolutely convergent by the definition of \tilde{D} .

Remark 2.3. When dealing with a bilinear form Q we will usually write $Q(u)$ instead of $Q(u, u)$.

We now introduce the *formal operator* \tilde{L} associated to \tilde{Q} . Let

$$\tilde{F} = \{u : V \rightarrow \mathbb{R} \mid \sum_{y \in V} b(x, y)|u(y)| < \infty \text{ for all } x \in V\}$$

and $m : V \rightarrow (0, \infty)$. We then define

$$\tilde{L} := \tilde{L}_{b, c, m} : \tilde{F} \rightarrow C(V)$$

via

$$(\tilde{L}u)(x) = \frac{1}{m(x)} \sum_{y \in V} b(x, y)(u(x) - u(y)) + \frac{c(x)}{m(x)}u(x).$$

The definition of \tilde{F} and (b2) ensure that the above sum is absolutely convergent. The following lemma is the crucial link between \tilde{L} and \tilde{Q} . Some version of it may be found in [HK](see [HKLW] and [KL] also).

Lemma 2.4 (Green's formula). *For all $u \in \tilde{D}$ and $v \in C_c(V)$ the equality*

$$\tilde{Q}(u, v) = \sum_{x \in V} (\tilde{L}u)(x)v(x)m(x) = \sum_{x \in V} u(x)(\tilde{L}v)(x)m(x)$$

holds.

Proof. The proof will be done in two steps. First, we show $\tilde{D} \subseteq \tilde{F}$ following Proposition 2.8 of [HKLW]. As a second step, we prove the desired equality as in Lemma 4.7 of [HK].

Step 1: Let $u \in \tilde{D}$. Then, using Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \sum_{y \in V} b(x, y) |u(y)| &\leq \sum_{y \in V} b(x, y) |u(x) - u(y)| + \sum_{y \in V} b(x, y) |u(x)| \\ &\leq \left(\sum_{y \in V} b(x, y) \right)^{1/2} \left(\sum_{y \in V} b(x, y) |u(x) - u(y)|^2 \right)^{1/2} + \deg(x) |u(x)| \\ &\leq \deg(x)^{1/2} \tilde{Q}(u)^{1/2} + \deg(x) |u(x)| < \infty. \end{aligned}$$

This shows $u \in \tilde{F}$.

Step 2: Let $u \in \tilde{D}$ and $v \in C_c(V)$. Step 1 and (b1) yield

$$\sum_{x, y \in V} b(x, y) |u(x)v(y)| = \sum_{y \in V} |v(y)| \sum_{x \in V} b(x, y) |u(x)| < \infty.$$

Moreover, by (b2) we obtain

$$\sum_{x, y \in V} b(x, y) |u(x)v(x)| = \sum_{x \in V} |v(x)| |u(x)| \sum_{y \in V} b(x, y) < \infty.$$

This allows us to rearrange the summation of

$$\sum_{x \in V} (\tilde{L}u)(x) v(x) m(x) = \sum_{x \in V} \left(\sum_{y \in V} b(x, y) (u(x) - u(y)) v(x) + c(x) u(x) v(x) \right)$$

and the statement follows by a simple computation. \square

The next lemma is standard (see e.g. Lemma 2.5 of [JP1]). It shows some continuity of differences of function values with respect to \tilde{Q} .

Lemma 2.5. *Let (b, c) be connected and $x, y \in V$. Then there exists a constant $K_{x, y} > 0$ such that for every $u \in \tilde{D}$*

$$|u(x) - u(y)| \leq K_{x, y} \tilde{Q}(u)^{1/2}$$

holds.

Proof. Let $x = x_0, \dots, x_n = y$ be a path connecting x and y such that the x_j are pairwise different. Set

$$K_{x, y} = \left(\sum_{j=1}^n \frac{1}{b(x_{j-1}, x_j)} \right)^{1/2}.$$

Then by Cauchy-Schwarz inequality we infer

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{j=1}^n |u(x_j) - u(x_{j-1})| \\ &\leq K_{x,y} \left(\sum_{j=1}^n b(x_{j-1}, x_j) |u(x_j) - u(x_{j-1})|^2 \right)^{1/2}. \end{aligned}$$

This yields the statement. \square

2.2 The Yamasaki space

Following [Soa1], we introduce a Hilbert space associated with a connected graph (b, c) . Fix $o \in V$. For $u, v \in \tilde{D}$ we can then define the inner product

$$\langle u, v \rangle_o = \tilde{Q}(u, v) + u(o)v(o).$$

Let $\|\cdot\|_o$ be the corresponding norm (it is nondegenerate since (b, c) is connected). The pair $(\tilde{D}, \langle \cdot, \cdot \rangle_o)$ will be called the Yamasaki space associated with (b, c) . We will write \mathbf{D} for short whenever we refer to \tilde{D} endowed with the topology generated by $\|\cdot\|_o$. Let us investigate some basic aspects of the structure of \mathbf{D} . The following proposition is part of Lemma 3.14 and Theorem 3.15 in [Soa1].

Proposition 2.6 (Properties of \mathbf{D}). *(a) For every $x \in V$ the functional $F_x : \mathbf{D} \rightarrow \mathbb{R}$, $u \mapsto u(x)$ is continuous with respect to $\|\cdot\|_o$. In particular, for $o, o' \in V$ the norms $\|\cdot\|_o$ and $\|\cdot\|_{o'}$ are equivalent.*
(b) \mathbf{D} equipped with $\langle \cdot, \cdot \rangle_o$ is a Hilbert space.

Proof. (a) We only need to prove the first statement. The 'In particular' part follows from the first, noting that $u \mapsto u(o)$ is continuous with respect to $\|\cdot\|_{o'}$.

Let $x \in V$ be given. Pick a path $o = x_0, \dots, x_n = x$ of pairwise different points from o to x and let K_{x_{i-1}, x_i} be constants for these vertices as in Lemma 2.5. Then

$$\begin{aligned} |u(x)| &\leq \sum_{i=1}^n |u(x_{i-1}) - u(x_i)| + |u(o)| \\ &\leq \tilde{Q}(u)^{1/2} \sum_{i=1}^n K_{x_{i-1}, x_i} + \|u\|_o. \end{aligned}$$

This finishes the proof of (a).

(b) It suffices to show completeness. Let (u_n) be a Cauchy sequence in \mathbf{D} . By (a) the sequence (u_n) converges pointwise towards a function u . Fatou's lemma yields

$$\tilde{Q}(u - u_n) \leq \liminf_{l \rightarrow \infty} \tilde{Q}(u_l - u_n),$$

which can be made small by choosing n large enough. This shows $u \in \mathbf{D}$ and $u_n \rightarrow u$ with respect to $\|\cdot\|_o$. \square

For our further discussion, we will need some more approximation results based on the following theorem. The presented proof was suggested by Daniel Lenz.

Theorem 1 (Convergence in \mathbf{D}). *Let $u_n, u \in \mathbf{D}$ be given. Then $u_n \xrightarrow{\|\cdot\|_o} u$ if and only if $\limsup \tilde{Q}(u_n) \leq \tilde{Q}(u)$ and $u_n(x) \rightarrow u(x)$ for every $x \in V$.*

Proof. The 'only if' part follows from Proposition 2.6 (a). Now, let u_n, u be given such that $u_n \rightarrow u$ pointwise and $\limsup \tilde{Q}(u_n) \leq \tilde{Q}(u)$. These two conditions imply that (u_n) is a bounded sequence in \mathbf{D} . Because \mathbf{D} is a Hilbert space every ball is weakly compact, thus every subsequence of (u_n) has a weakly convergent subsequence. Since (u_n) converges pointwise towards u , all the occurring limits must coincide. We infer $u_n \rightarrow u$ weakly in \mathbf{D} . Furthermore

$$\begin{aligned} 0 &\leq \tilde{Q}(u - u_n) + (u(o) - u_n(o))^2 \\ &= \tilde{Q}(u) + u(o)^2 + \tilde{Q}(u_n) + u_n(o)^2 - 2\langle u, u_n \rangle_o, \end{aligned}$$

which yields the statement after taking \limsup . \square

Corollary 2.7. *Let $u \in \mathbf{D}$. Then, for any natural number N the function $u_N = ((-N) \vee u) \wedge N$ belongs to \mathbf{D} and $u_N \xrightarrow{\|\cdot\|_o} u$ as $N \rightarrow \infty$.*

Proof. Lemma 2.1 shows $\tilde{Q}(u_N) \leq \tilde{Q}(u)$. Now we infer the statement by the above theorem. \square

Corollary 2.8. *Let \mathbf{D} be associated to a graph $(b, 0)$ and (e_n) a sequence in \mathbf{D} such that*

$$\|e_n - 1\|_o \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Furthermore, let $u \in \mathbf{D}$ with $0 \leq u \leq 1$ be given. Then

$$\|e_n \wedge u - u\|_o \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof. Proposition 2.6 shows that convergence with respect to $\|\cdot\|_o$ implies pointwise convergence, hence $e_n \rightarrow 1$ pointwise. Since $0 \leq u \leq 1$ this also yields $e_n \wedge u \rightarrow u$ pointwise. Using Theorem 1 it suffices to show $\limsup \tilde{Q}(e_n \wedge u) \leq \tilde{Q}(u)$ to obtain the statement. Let us observe

$$\begin{aligned} \tilde{Q}(e_n \wedge u)^{1/2} &= \frac{1}{2} \tilde{Q}(u + e_n - |u - e_n|)^{1/2} \\ &\leq \frac{1}{2} [\tilde{Q}(u)^{1/2} + \tilde{Q}(e_n)^{1/2} + \tilde{Q}(|u - e_n|)^{1/2}] \\ &\leq \tilde{Q}(u)^{1/2} + \tilde{Q}(e_n)^{1/2}. \end{aligned}$$

Because $c \equiv 0$, $\|e_n - 1\|_o \rightarrow 0$ implies $\tilde{Q}(e_n) \rightarrow 0$. Thus the above calculation yields the statement after taking \limsup . \square

2.3 Dirichlet forms

Let m be a measure of full support on V , i.e. a function $m : V \rightarrow (0, \infty)$. Then the sets

$$\ell^p(V, m) = \{u : V \rightarrow \mathbb{R} \mid \sum_{x \in V} |u(x)|^p m(x) < \infty\}$$

endowed with the norm

$$\|u\|_p = \left(\sum_{x \in V} |u(x)|^p m(x) \right)^{1/p}$$

are Banach spaces. Moreover, the case $p = 2$ provides a Hilbert space with inner product given by

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x)m(x).$$

As usual, $\ell^\infty(V)$ denotes the set of all bounded functions on V . It comes with the corresponding norm

$$\|u\|_\infty = \sup_{x \in V} |u(x)|.$$

In the subsequent chapters we will be concerned with restrictions of \tilde{Q} to certain ℓ^2 -domains such that the emerging forms become Dirichlet forms. There appear to be two extremal choices for such domains. The first one is the maximal ℓ^2 -domain given by $D(Q^{(N)}) = \tilde{D} \cap \ell^2(V, m)$. We will call $\tilde{Q}|_{D(Q^{(N)})}$ the Neumann form associated to (b, c) and write $Q^{(N)}$ for short. Then the following is true.

Proposition 2.9. $Q^{(N)}$ is a Dirichlet form. Its associated operator $L^{(N)}$ is a restriction of \tilde{L} with domain $D(L^{(N)})$ satisfying

$$D(L^{(N)}) \subseteq \{u \in \tilde{D} \cap \ell^2(V, m) \mid \tilde{L}u \in \ell^2(V, m)\}.$$

Proof. For the first statement it suffices to show the Markov property of $Q^{(N)}$ (see appendix for the definition of a Dirichlet form). Let $C : \mathbb{R} \rightarrow \mathbb{R}$, such that $C(0) = 0$ and $|C(x) - C(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$ be given. Lemma 2.1 shows that for $u \in \tilde{D}$

$$\tilde{Q}(C \circ u) \leq \tilde{Q}(u)$$

holds. If furthermore $u \in \ell^2(V, m)$ the function $C \circ u$ also belongs to $\ell^2(V, m)$. This proves the Markov property.

Let us now turn our attention towards the statement about the corresponding operator. Suppose $u \in D(L^{(N)})$. Then, using the notation $\hat{\delta}_x = m(x)^{-1}\delta_x$ and Lemma 2.4 we obtain

$$(L^{(N)}u)(x) = \langle L^{(N)}u, \hat{\delta}_x \rangle = Q^{(N)}(u, \hat{\delta}_x) = \langle \tilde{L}u, \hat{\delta}_x \rangle = (\tilde{L}u)(x).$$

This shows that $L^{(N)}$ is a restriction of \tilde{L} and also implies

$$D(L^{(N)}) \subseteq \{u \in \tilde{D} \cap \ell^2(V, m) \mid \tilde{L}u \in \ell^2(V, m)\}.$$

□

The second important choice for a domain is given by

$$D(Q^{(D)}) = \overline{C_c(V)}^{\|\cdot\|_Q},$$

where the closure is taken with respect to the form norm

$$\|\cdot\|_Q = \sqrt{\tilde{Q}(\cdot) + \|\cdot\|_2^2},$$

in $\ell^2(V, m) \cap \tilde{D}$. We will call $\tilde{Q}|_{D(Q^{(D)})}$ the regular Dirichlet form associated to (b, c) and denote it by $Q^{(D)}$. It might be thought of being the minimal closed ℓ^2 -restriction of \tilde{Q} containing $C_c(V)$.

Proposition 2.10. $Q^{(D)}$ is a regular Dirichlet form. Its associated operator $L^{(D)}$ is a restriction of \tilde{L} with domain $D(L^{(D)})$ satisfying

$$D(L^{(D)}) = \{u \in D(Q^{(D)}) \mid \tilde{L}u \in \ell^2(V, m)\}.$$

Proof. For the statement about $Q^{(D)}$ it again suffices to show the Markov property. Let C be a normal contraction and $u \in D(Q^{(D)})$. We need to show $C \circ u \in D(Q^{(D)})$. By the definition of $D(Q^{(D)})$ there exists a sequence $(u_n) \subseteq C_c(V)$ converging towards u with respect to $\|\cdot\|_Q$. Obviously $\|C \circ u_n - C \circ u\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Furthermore,

$$\limsup_{n \rightarrow \infty} \tilde{Q}(C \circ u_n) \leq \limsup_{n \rightarrow \infty} \tilde{Q}(u_n) = \tilde{Q}(u).$$

Therefore, $C \circ u_n$ is a bounded sequence in the Hilbert space $(\tilde{D} \cap \ell^2(V, m), \|\cdot\|_Q)$. By the weak compactness of balls in Hilbert spaces we conclude that any of its subsequences has a weakly convergent subsequence. Because of the ℓ^2 -convergence of $C \circ u_n$ all the occurring limits must coincide with $C \circ u$. We infer $C \circ u_n \rightarrow C \circ u$ weakly in $(\tilde{D} \cap \ell^2(V, m), \|\cdot\|_Q)$. This shows that $C \circ u$ belongs to the weak closure of $C_c(V)$ in $(\tilde{D} \cap \ell^2(V, m), \|\cdot\|_Q)$. Because $C_c(V)$ is convex, this weak closure coincides with the closure of $C_c(V)$ with respect to $\|\cdot\|_Q$. Therefore $C \circ u \in D(Q^{(D)})$.

Let us prove the statement about the corresponding operator $L^{(D)}$. Just as in the proof of the previous proposition, we can show that $L^{(D)}$ is a restriction of \tilde{L} and

$$D(L^{(D)}) \subseteq \{u \in D(Q^{(D)}) \mid \tilde{L}u \in \ell^2(V, m)\}.$$

Now let $v \in \{u \in D(Q^{(D)}) \mid \tilde{L}u \in \ell^2(V, m)\}$ be given. It remains to show $v \in D(L^{(D)})$. By the correspondence $Q^{(D)} \leftrightarrow L^{(D)}$ (see appendix) it is sufficient to confirm the validity of

$$Q^{(D)}(v, w) = \langle \tilde{L}v, w \rangle$$

for all $w \in D(Q^{(D)})$. From Lemma 2.4 we infer that the above equality is true for all $w \in C_c(V)$. Since $C_c(V)$ is dense in $D(Q^{(D)})$ with respect to $\|\cdot\|_Q$, it extends to all $w \in D(Q^{(D)})$. This finishes the proof. \square

Remark 2.11. • *The above proof and the one of Theorem 1 use a similar argument.*

They show that a bounded sequence in a Hilbert space that is convergent in some 'weak' sense (pointwise, in ℓ^2 , ...) is already weakly convergent in that Hilbert space.

- *There seems to be no explicit proof for $Q^{(D)}$ being a Dirichlet form in the literature. Usually this property is deduced from Theorem 3.1.1. of [FOT] which uses general principles.*
- *The above characterization of the domain of $L^{(D)}$ seems to be new.*

Now we can introduce the main objects of our studies.

Definition 2.12 (Dirichlet form associated with (b, c)). *Let (b, c) be a graph over V . A Dirichlet form Q is called associated to (b, c) if it is a restriction of $\tilde{Q}_{b,c}$ and its domain $D(Q)$ satisfies*

$$D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)}).$$

Remark 2.13. *The Dirichlet forms considered above seem to be a special class of examples of Dirichlet forms on $\ell^2(V, m)$. However, as seen in [KL], it turns out that every regular Dirichlet form on a discrete space coincides with a regular Dirichlet form associated to a graph (b, c) (see appendix). In this sense the forms $Q^{(D)}$ are exactly the regular Dirichlet forms on discrete spaces.*

Chapter 3

General theory

In the following we study recurrence and transience of Dirichlet forms associated to a graph (b, c) by using the theory presented in [FOT]. The discrete structure of the underlying ℓ^2 -space allows us to simplify many technical details. Therefore, some definitions and statements differ slightly from the ones found in [FOT]. At first, we introduce two notions of transience (see Definition 3.1 and Definition 3.4) and show that they coincide (Theorem 2). Afterwards, we extend the Dirichlet forms to their extended space $D(Q)_e$ (Definition 3.6) and provide characterizations of recurrence and transience in terms of properties of this space (see Theorem 3 and Theorem 4).

Let Q be a Dirichlet form associated to a graph (b, c) and let e^{-tL} be its associated (strongly continuous) semigroup (see appendix for a discussion of Dirichlet forms and their associated objects). Let us introduce the notions of recurrence and transience.

Definition 3.1 (Transient semigroup). *The semigroup e^{-tL} is called transient if for every $x, y \in V$*

$$G(x, y) := \int_0^\infty e^{-tL} \delta_x(y) dt < \infty.$$

It is called recurrent if $G(x, y) = \infty$ holds for all $x, y \in V$.

The next proposition shows the dichotomy of the above concepts whenever the graph (b, c) is connected.

Proposition 3.2. *Let (b, c) be connected. The semigroup e^{-tL} is transient if and only if there exist some $x, y \in V$, such that $G(x, y) < \infty$. In particular e^{-tL} is either recurrent or transient.*

Proof. Let $x, y \in V$, such that $G(x, y) < \infty$ and let $w, z \in V$ arbitrary. We need to show $G(w, z) < \infty$. The functions $\tilde{\delta}_v = m(v)^{-1/2}\delta_v$ form an orthonormal basis in $\ell^2(V, m)$. Using the semigroup property and that e^{-tL} is positivity preserving (see appendix, property (S5)) we then obtain for $t > 1$

$$\begin{aligned}
 (e^{-tL}\delta_x)(y) &= (e^{-(t-1)L}e^{-L}\delta_x)(y) \\
 &= \left[e^{-(t-1)L} \left(\sum_{v \in V} \langle e^{-L}\delta_x, \tilde{\delta}_v \rangle \tilde{\delta}_v \right) \right] (y) \\
 &= \frac{1}{m(y)} \sum_{v \in V} \langle e^{-L}\delta_x, \tilde{\delta}_v \rangle \langle e^{-(t-1)L}\tilde{\delta}_v, \delta_y \rangle \\
 &\geq \frac{1}{m(y)} \langle e^{-L}\delta_x, \tilde{\delta}_w \rangle \langle e^{-(t-1)L}\tilde{\delta}_w, \delta_y \rangle.
 \end{aligned} \tag{3.1}$$

Since (b, c) is connected, Theorem A.6 shows e^{-tL} is positivity improving, hence

$$\langle e^{-L}\delta_x, \tilde{\delta}_w \rangle > 0.$$

We can now conclude the finiteness of $G(w, y)$ by integrating both sides of inequality (3.1) from 1 to ∞ . A similar computation shows the finiteness of $G(w, z)$. Now the 'In particular' part is an immediate consequence of the previous. □

The next proposition shows how transience of the semigroup is related to the resolvent $(L + \alpha)^{-1}$ associated with Q .

Proposition 3.3. *For all $x, y \in V$ the equality*

$$\int_0^\infty e^{-tL}\delta_x(y)dt = \lim_{\alpha \rightarrow 0+} (L + \alpha)^{-1}\delta_x(y)$$

holds.

Proof. By monotone convergence we infer

$$\int_0^\infty e^{-tL}\delta_x(y)dt = \lim_{\alpha \rightarrow 0+} \int_0^\infty e^{-t\alpha}e^{-tL}\delta_x(y)dt.$$

Let μ be the spectral measure associated with L such that

$$\frac{1}{m(y)} \langle e^{-tL}\delta_x, \delta_y \rangle = \int_0^\infty e^{-t\lambda} d\mu(\lambda).$$

Since $e^{-t(\alpha+\lambda)}$ is integrable in t on $[0, \infty)$ and μ is of bounded variation, we can use Fubini's theorem to obtain

$$\begin{aligned} \int_0^\infty e^{-t\alpha} e^{-tL} \delta_x(y) dt &= \int_0^\infty \int_0^\infty e^{-t\alpha} e^{-t\lambda} d\mu(\lambda) dt \\ &= \int_0^\infty \int_0^\infty e^{-t(\alpha+\lambda)} dt d\mu(\lambda) \\ &= \int_0^\infty \frac{1}{\lambda + \alpha} d\mu(\lambda) \\ &= (L + \alpha)^{-1} \delta_x(y). \end{aligned}$$

This finishes the proof. \square

Definition 3.4 (Transient Dirichlet form). *The Dirichlet form Q is called transient if there exists a strictly positive $g \in \ell^1(V, m) \cap \ell^\infty(V)$, such that*

$$\sum_{x \in V} |u(x)| g(x) m(x) \leq \sqrt{Q(u)},$$

for all $u \in D(Q)$. Such a function g is called reference function for Q .

We already know that e^{-tL} is a family of bounded operators on $\ell^2(V, m)$, strongly continuous in t . Thus $t \mapsto e^{-tL} f(x)$ is continuous for any $f \in \ell^2(V, m)$ and any $x \in V$. We use this to introduce the 0-th order resolvent operator

$$G : \ell_+^2(V, m) \rightarrow \{u : V \rightarrow [0, \infty]\}$$

acting as

$$(Gf)(x) = \int_0^\infty e^{-tL} f(x) dt.$$

Here $\ell_+^2(V, m)$ denotes the set of all non-negative ℓ^2 -functions. Note that $(Gf)(x)$ may take the value ∞ . It is obvious, that $G(x, y) = (G\delta_x)(y)$. The following lemma will allow us to prove the equivalence of Definition 3.1 and 3.4.

Lemma 3.5. *Let $g \in \ell^1(V, m) \cap \ell^2(V, m)$ be nonnegative. Then,*

$$\sup_{u \in D(Q)} \frac{\langle |u|, g \rangle}{\sqrt{Q(u)}} = \sqrt{\langle g, Gg \rangle}.$$

In particular $\sup_{u \in D(Q)} \frac{\langle |u|, g \rangle}{\sqrt{Q(u)}}$ is finite if and only if $\langle g, Gg \rangle$ is finite.

Proof. For $f \in \ell^2(V, m)$ and $x \in V$ we will use the notation

$$(S_t f)(x) = \int_0^t e^{-sL} f(x) ds.$$

With the help of Theorem A.7 we conclude

$$\|S_t f\|_2 \leq \int_0^t \|e^{-sL} f\|_2 ds \leq t \|f\|_2.$$

Therefore, $S_t f$ is a bounded linear operator on $\ell^2(V, m)$. Now the proof will be done in three steps. The 'In particular' part will be an immediate consequence of Step 2 and Step 3.

Step 1: For $f \in \ell^2(V, m)$ we prove the identity

$$Q(S_t f, u) = \langle f - e^{-tL} f, u \rangle \quad (3.2)$$

by showing that $S_t f \in D(L)$ and $LS_t f = f - e^{-tL} f$. Using the correspondence $L \leftrightarrow e^{-tL}$ we need to compute the derivative of $s \mapsto -e^{-sL} S_t f$ at 0 as a strong limit (see appendix). Theorem A.8 yields

$$\begin{aligned} \frac{S_t f - e^{-sL} S_t f}{s} &= \frac{S_t f - S_t e^{-sL} f}{s} \\ &= \frac{1}{s} \left(\int_0^s e^{-uL} f du - \int_t^{s+t} e^{-uL} f du \right). \end{aligned}$$

With the help of Theorem A.7 and a mean value theorem for Riemann integrals we compute

$$\begin{aligned} \left\| \frac{1}{s} \int_0^s e^{-uL} f du - f \right\|_2 &= \left\| \frac{1}{s} \int_0^s e^{-uL} f - f du \right\|_2 \\ &\leq \frac{1}{s} \int_0^s \|e^{-uL} f - f\|_2 du \\ &= \|e^{-\theta L} f - f\|_2 \end{aligned}$$

where $\theta \in (0, s)$. Now we can take the limit $s \rightarrow 0$ to obtain

$$\frac{1}{s} \int_0^s e^{-uL} f du \rightarrow f.$$

An analogue computation shows

$$\frac{1}{s} \int_t^{s+t} e^{-uL} f du \rightarrow e^{-tL} f \text{ as } s \rightarrow 0.$$

This implies formula (3.2).

Step 2: Let us assume

$$\sup_{u \in D(Q)} \frac{\langle |u|, g \rangle}{\sqrt{Q(u)}} = c < \infty.$$

We show $\sqrt{\langle g, Gg \rangle} \leq c$. Since $e^{-tL}g \geq 0$ and $S_tg \geq 0$, we obtain

$$\langle S_tg, g \rangle \leq c\sqrt{Q(S_tg)} \stackrel{(3.2)}{=} c\sqrt{\langle g - e^{-tL}g, S_tg \rangle} \leq c\sqrt{\langle g, S_tg \rangle}.$$

This implies $\sqrt{\langle S_tg, g \rangle} \leq c$. Using monotone convergence we can take the limit $t \rightarrow \infty$ to obtain $\sqrt{\langle g, Gg \rangle} \leq c$.

Step 3: Suppose that $\langle g, Gg \rangle < \infty$. By monotone convergence and semigroup properties, we conclude

$$\begin{aligned} \langle g, Gg \rangle &= \int_0^\infty \langle e^{-tL}g, g \rangle dt \\ &= \int_0^\infty \langle e^{-t/2L}g, e^{-t/2L}g \rangle dt. \end{aligned}$$

Furthermore $\langle e^{-t/2L}g, e^{-t/2L}g \rangle$ is nonincreasing in t (use semigroup property and $\|e^{-sL}\| \leq 1$). Thus the finiteness of the above integral yields

$$\lim_{t \rightarrow \infty} \langle e^{-tL}g, g \rangle = \lim_{t \rightarrow \infty} \langle e^{-t/2L}g, e^{-t/2L}g \rangle = 0.$$

On account of (3.2) we obtain

$$\begin{aligned} \langle |u|, g \rangle &\stackrel{(3.2)}{=} Q(S_tg, |u|) + \langle |u|, e^{-tL}g \rangle \\ &\leq \sqrt{Q(S_tg)}\sqrt{Q(|u|)} + \sqrt{\langle e^{-tL}g, e^{-tL}g \rangle}\sqrt{\langle u, u \rangle} \\ &\stackrel{(3.2)}{=} \sqrt{\langle g - e^{-tL}g, S_tg \rangle}\sqrt{Q(|u|)} + \sqrt{\langle e^{-tL}g, e^{-tL}g \rangle}\sqrt{\langle u, u \rangle} \\ &\leq \sqrt{\langle S_tg, g \rangle}\sqrt{Q(u)} + \sqrt{\langle e^{-2tL}g, g \rangle}\sqrt{\langle u, u \rangle}. \end{aligned}$$

Now the inequality $\langle |u|, g \rangle \leq \sqrt{\langle g, Gg \rangle}\sqrt{Q(u)}$ follows by taking the limit $t \rightarrow \infty$ in the above computations. \square

The next theorem shows that both concepts of transience coincide.

Theorem 2. *The Dirichlet form Q is transient if and only if its associated semigroup e^{-tL} is transient.*

Proof. Assume Q is transient and let $g \in \ell^1(V, m) \cap \ell^\infty(V) \subseteq \ell^2(V, m)$ be a reference function of Q . On account of

$$\sum_{x \in V} |u(x)|g(x)m(x) \leq \sqrt{Q(u)} \text{ for } u \in D(Q),$$

Lemma 3.5 shows that $\langle g, Gg \rangle \leq 1$ must hold. Since reference functions are strictly positive this implies $Gg(x) < \infty$ for all $x \in V$. Therefore the non-negativity of $e^{-tL}\delta_x$ and the self-adjointness of e^{-tL} yield

$$\begin{aligned} \int_0^\infty e^{-tL}\delta_x(y)dt &= \frac{1}{g(y)m(y)} \int_0^\infty \langle e^{-tL}\delta_x, g(y)\delta_y \rangle dt \\ &\leq \frac{1}{g(y)m(y)} \int_0^\infty \langle e^{-tL}\delta_x, g \rangle dt \\ &= \frac{m(x)}{g(y)m(y)} \int_0^\infty e^{-tL}g(x)dt < \infty. \end{aligned}$$

Now assume that e^{-tL} is transient, i.e. $G(x, y) = (G\delta_x)(y) < \infty$ for all $x, y \in V$. By applying Lemma 3.5 to δ_x , we conclude

$$\sup_{u \in D(Q)} \frac{|u(x)|m(x)}{\sqrt{Q(u)}} = G(x, x)m(x).$$

Since $D(Q)$ is dense in $\ell^2(V, m)$, for every $x \in V$ there exists a function $v_x \in D(Q)$ such that $|v_x(x)| > 0$. Therefore the above equation shows $G(x, x) > 0$ for every $x \in V$. We need to find a reference function g as in Definition 3.4. Let us define g as

$$g(x) = \frac{a_x}{G(x, x)m(x)},$$

where (a_x) is a sequence of strictly positive numbers, chosen such that g belongs to $\ell^1(V, m) \cap \ell^\infty(V)$ and

$$\sum_{x \in V} a_x = 1.$$

We obtain for $u \in D(Q)$

$$\begin{aligned} \sum_{x \in V} |u(x)|g(x)m(x) &= \sum_{x \in V} \frac{|u(x)|a_x}{G(x, x)} \\ &\leq \sum_{x \in V} a_x \sqrt{Q(u)} \\ &= \sqrt{Q(u)}, \end{aligned}$$

as was to be shown. □

As a next step we introduce the extended Dirichlet space $D(Q)_e$ associated with Q . It will turn out that investigations about $D(Q)_e$ lead to characterizations of transience.

Definition 3.6 (Extended Dirichlet space). *We call the set*

$$D(Q)_e = \{u : V \rightarrow \mathbb{R} \mid \exists Q\text{-Cauchy sequence } (u_n) \subseteq D(Q) \text{ s.t. } u_n \rightarrow u \text{ pointwise}\},$$

the extended Dirichlet space of Q . For $u \in D(Q)_e$ we say a sequence (u_n) as in the above set is an approximating sequence for u .

We want to extend the Dirichlet form Q to its extended space. This may be done via the next lemma.

Lemma 3.7. *Let $u \in D(Q)_e$ and $(u_n) \subseteq D(Q)$ be an approximating sequence for u . Then $u \in \tilde{D}$ and*

$$\lim_{n \rightarrow \infty} \tilde{Q}(u - u_n) = 0.$$

Proof. By Fatou's lemma we obtain

$$\begin{aligned} \tilde{Q}(u - u_n)^{1/2} &\leq \liminf_{l \rightarrow \infty} \tilde{Q}(u_l - u_n)^{1/2} \\ &= \liminf_{l \rightarrow \infty} Q(u_l - u_n)^{1/2}. \end{aligned}$$

Since (u_n) is a Q -Cauchy sequence, this yields $u \in \tilde{D}$ and $\lim \tilde{Q}(u - u_n) = 0$. \square

For $u, v \in D(Q)_e$ let us set

$$Q(u, v) := \tilde{Q}(u, v)$$

to extend Q to $D(Q)_e$. Then the above lemma shows that $D(Q)$ is dense in $D(Q)_e$ with respect to the pseudometric induced by Q on $D(Q)_e$. Whenever the underlying graph (b, c) is connected, we can compute $D(Q)_e$ in the following way.

Proposition 3.8. *Let (b, c) be connected. The extended Dirichlet space of a Dirichlet form Q associated to (b, c) is given by the closure of $D(Q)$ in \mathbf{D} , i.e.*

$$D(Q)_e = \overline{D(Q)}^{\|\cdot\|_o}.$$

Proof. Let $u \in D(Q)_e$ be given and $(u_n) \subseteq D(Q)$ be an approximating sequence for u . Then (u_n) is a Cauchy sequence with respect to $\|\cdot\|_o$. Since \mathbf{D} is complete (u_n) converges in \mathbf{D} to some function v . By the pointwise convergence of the u_n to u we infer $u = v$. This shows $u \in \overline{D(Q)}^{\|\cdot\|_o}$. The other inclusion follows from Proposition 2.6. \square

We turn our studies towards criteria for transience. Let us at first observe a necessary condition.

Lemma 3.9. *Suppose Q is transient. Then $(D(Q)_e, Q)$ is a Hilbert space.*

Proof. By transience there exists a strictly positive $g \in \ell^1(V, m) \cap \ell^\infty(V)$ such that

$$\langle |u|, g \rangle \leq \sqrt{Q(u)}, \quad (3.3)$$

for all $u \in D(Q)$. By the definition of $D(Q)_e$ and Lemma 3.7 this inequality extends to all $u \in D(Q)_e$, thus Q is non-degenerate on $D(Q)_e$. Let us turn our attention to completeness. Let (u_n) be a Cauchy sequence in $(D(Q)_e, Q)$. Then (3.3) implies pointwise convergence of (u_n) to a function u . This yields $u \in D(Q)_e$ with approximating sequence (u_n) . We then infer by Lemma 3.7

$$Q(u_n - u) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

This finishes the proof. □

Below we want to prove characterizations for recurrence and transience. In the recurrent case an important step will be to construct a sequence of functions in $D(Q)$ converging to 1 with respect to Q . To obtain such a sequence we will need to study a transient Dirichlet form Q^g , which is defined by perturbing Q in the following way. Let $g \in \ell^1(V, m) \cap \ell^\infty(V)$ be strictly positive. We then set

$$\begin{aligned} Q^g : D(Q) \times D(Q) &\rightarrow \mathbb{R} \\ Q^g(u, v) &= Q(u, v) + \langle gu, v \rangle. \end{aligned}$$

It is immediate that Q^g is a Dirichlet form on $\ell^2(V, m)$. Let us denote the associated self-adjoint operator by L^g .

Proposition 3.10. *Q^g is a transient Dirichlet form.*

Proof. Let us calculate

$$\sum_{x \in V} |u(x)|g(x)m(x) \leq \sqrt{\|g\|_1} \sqrt{\langle gu, u \rangle} \leq \sqrt{\|g\|_1} \sqrt{Q^g(u)}.$$

Thus transience holds by definition with $\tilde{g} = g/\sqrt{\|g\|_1}$ as a reference function. □

As mentioned before, the following lemma is the key ingredient for characterizing recurrence.

Lemma 3.11. *Let Q be recurrent. Then for any $x \in V$*

$$\lim_{\alpha \rightarrow 0+} (L^g + \alpha)^{-1} g(x) = 1.$$

Proof. Let us write Q_α for the perturbed form

$$Q_\alpha(u, v) = Q(u, v) + \alpha \langle u, v \rangle.$$

Recall that

$$Q_\alpha(w, u) = \langle f, u \rangle$$

for all $u \in D(Q)$ if and only if $w = (L + \alpha)^{-1} f$ (see appendix). Now the proof will be done in three steps.

Step 1: We compute the resolvent of Q^g in terms of the resolvent of Q . Using the above characterization of the resolvent with respect to Q_α let us observe that for any $f \in \ell^2(V, m)$ and $u \in D(Q)$

$$\begin{aligned} Q_\alpha((L^g + \alpha)^{-1} f, u) &= Q_\alpha^g((L^g + \alpha)^{-1} f, u) - \langle g(L^g + \alpha)^{-1} f, u \rangle \\ &= \langle f - g(L^g + \alpha)^{-1} f, u \rangle. \end{aligned}$$

This implies

$$(L^g + \alpha)^{-1} f = (L + \alpha)^{-1} [f - g(L^g + \alpha)^{-1} f]. \quad (3.4)$$

Step 2: We show $(L^g + \alpha)^{-1} g \leq 1$ by considering Q as a Dirichlet form on $\ell^2(V, m')$ for a certain measure m' different from m . For $\alpha > 0$ the form Q is a Dirichlet form on $\ell^2(V, (g + \alpha)m)$. Let $f \in \ell^2(V, (g + \alpha)m)$. We then obtain for any $v \in D(Q)$

$$\begin{aligned} &Q((L^g + \alpha)^{-1} [\alpha f + fg], v) + \langle (L^g + \alpha)^{-1} [\alpha f + fg], v \rangle_{(g+\alpha)m} \\ &= Q^g((L^g + \alpha)^{-1} [\alpha f + fg], v) + \alpha \langle (L^g + \alpha)^{-1} [\alpha f + fg], v \rangle \\ &= Q_\alpha^g((L^g + \alpha)^{-1} [\alpha f + fg], v) \\ &= \langle \alpha f + gf, v \rangle = \langle f, v \rangle_{(\alpha+g)m}. \end{aligned}$$

From the above calculation we infer that $(L^g + \alpha)^{-1} [\alpha f + fg]$ is the 1-order resolvent of f associated to Q as a Dirichlet form on $\ell^2(V, (g + \alpha)m)$. Now the Markov property of this

resolvent (see appendix) implies that for any $f \in \ell^2(V, (g + \alpha)m)$ satisfying $0 \leq f \leq 1$ the inequality

$$(L^g + \alpha)^{-1}[\alpha f + fg] \leq 1 \quad (3.5)$$

holds. Pick a sequence (f_n) in $\ell^2(V, (g + \alpha)m)$ such that $0 \leq f_n \leq f_{n+1} \leq 1$ and $\lim f_n(x) = 1$ for all $x \in V$. Then $f_n g \rightarrow g$ in $\ell^2(V, m)$. Now inequality 3.5 together with the non-negativity of $(L^g + \alpha)^{-1}(\alpha f_n)$ yields

$$\begin{aligned} (L^g + \alpha)^{-1}g(x) &= \lim_{n \rightarrow \infty} (L^g + \alpha)^{-1}(f_n g)(x) \\ &\leq \limsup_{n \rightarrow \infty} [1 - (L^g + \alpha)^{-1}(\alpha f_n)(x)] \\ &\leq 1. \end{aligned}$$

Step 3: Equation (3.4) applied to g and Step 2 imply

$$\begin{aligned} 1 &\geq \lim_{\alpha \rightarrow 0+} (L^g + \alpha)^{-1}g(x) \\ &= \lim_{\alpha \rightarrow 0+} (L + \alpha)^{-1} [g - g(L^g + \alpha)^{-1}g] (x) \\ &= \lim_{\alpha \rightarrow 0+} \frac{1}{m(x)} \langle g(1 - (L^g + \alpha)^{-1}g), (L + \alpha)^{-1}\delta_x \rangle \\ &\geq \limsup_{\alpha \rightarrow 0+} \frac{g(x)(1 - (L^g + \alpha)^{-1}g(x))}{m(x)} \langle \delta_x, (L + \alpha)^{-1}\delta_x \rangle \geq 0. \end{aligned}$$

Since $(L + \alpha)^{-1}\delta_x(x)$ converges to infinity by our assumptions (Q was supposed to be recurrent), $(L^g + \alpha)^{-1}g(x)$ must tend to one. This finishes the proof. \square

Now we can prove the main theorems of this chapter.

Theorem 3 (Abstract characterization of transience). *Let (b, c) be connected and Q a Dirichlet form associated with (b, c) . Then the following conditions are equivalent:*

- (i) e^{-tL} is transient.
- (ii) $u = 0$ for every $u \in D(Q)_e$ with $Q(u) = 0$.
- (iii) $(D(Q)_e, Q)$ is a real Hilbert space.

Proof. '(i) \Rightarrow (iii)': This has already been shown.

'(iii) \Rightarrow (ii)': This is trivial.

'(ii) \Rightarrow (i)': Assume e^{-tL} is recurrent. Then, for a strictly positive $g \in \ell^1(V, m) \cap \ell^\infty(V)$, Lemma 3.11 implies $u_n := (L^g + 1/n)^{-1}g \rightarrow 1$ pointwise. Furthermore, by the correspondence $(L^g + \alpha)^{-1} \leftrightarrow Q^g$ we obtain

$$Q(u_n, u_n) = \langle g, u_n \rangle - \frac{1}{n} \langle u_n, u_n \rangle - \langle gu_n, u_n \rangle.$$

Because $g \in \ell^1(V, m)$ and u_n is uniformly bounded by 1 (see step 2 of the previous proof), we infer with Lebesgue's theorem

$$\lim_{n \rightarrow \infty} \langle g, u_n \rangle = \lim_{n \rightarrow \infty} \langle gu_n, u_n \rangle = \sum_{x \in V} g(x)m(x).$$

This shows

$$Q(u_n, u_n) \leq \langle g, u_n \rangle - \langle gu_n, u_n \rangle \rightarrow 0, \text{ as } n \rightarrow \infty.$$

The above computations imply $1 \in D(Q)_e$ and Lemma 3.7 yields $Q(1) = 0$. This is a contradiction to (ii) which shows that e^{-tL} is not recurrent. Since (b, c) is connected, this implies transience. \square

Theorem 4 (Abstract characterization of recurrence). *Let (b, c) be connected and Q a Dirichlet form associated with (b, c) . Then the following conditions are equivalent:*

- (i) e^{-tL} is recurrent.
- (ii) There exists a sequence (u_n) in $D(Q)$ satisfying $\lim u_n = 1$ pointwise and $\lim Q(u_n) = 0$.
- (iii) $1 \in D(Q)_e$ and $Q(1) = 0$.

Proof. The implication '(i) \Rightarrow (ii)' has already been shown in the proof of Theorem 3.

'(ii) \Rightarrow (iii)' follows from the definition of $D(Q)_e$ and Lemma 3.7.

'(iii) \Rightarrow (i)': By the second statement of Theorem 3, (iii) implies non-transience of e^{-tL} . Since (b, c) is connected this yields recurrence. \square

As a consequence of these theorems, let us note that a nonvanishing potential c implies transience.

Proposition 3.12. *Let (b, c) be a connected graph such that $c \not\equiv 0$. Assume Q is a Dirichlet form associated with (b, c) . Then Q is transient.*

Proof. If 1 belonged to $D(Q)_e$, we would obtain

$$Q(1) = \tilde{Q}(1) = \sum_{x \in V} c(x) \neq 0.$$

Then Theorem 4 implies that Q is not recurrent since condition (iii) would fail. Because (b, c) is connected, we infer the transience of Q .

□

Remark 3.13. *In light of the above result we will assume $c \equiv 0$ in the subsequent text whenever we deal with recurrence/transience.*

Chapter 4

Discrete time v.s. continuous time

In this chapter we are going to compare the notion of recurrence of a graph $(b, 0)$ which is commonly used in several textbooks (like in [Soa1] and [Woe]) with the one developed in the previous chapter. First we introduce a random walk associated with $(b, 0)$ and then explain what is usually meant by recurrence of this graph (see Definition 4.1). Afterwards we give a characterization of this in terms of the structure of the transition matrix of the random walk (Theorem 5) which is taken from [Woe]. Finally we prove that recurrence of $(b, 0)$ is equivalent to recurrence of the regular Dirichlet form $Q^{(D)}$ provided that $m \equiv 1$ (see Theorem 6). A short discussion about the stochastic interpretation of these results concludes this chapter.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A sequence of random variables $(X_n)_{n \geq 0}$ on Ω with values in V will be called a random walk associated with $(b, 0)$ if the following conditions are satisfied:

$$\mathbb{P}(X_{n+1} = y \mid X_n = x) = \frac{b(x, y)}{\deg(x)} \text{ for all } x, y \in V, n \geq 0$$

and

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

for all $n \geq 0$ and $x_0, \dots, x_{n+1} \in V$. We will skip the prove of the existence of such a Markov chain since this is standard.

Given a random walk associated with $(b, 0)$ one might ask about the long time behaviour of $(X_n)_{n \geq 0}$. In particular the question whether $(X_n)_{n \geq 0}$ returns to one particular point infinitely often is of interest.

Definition 4.1 (Recurrence of a random walk). A random walk $(X_n)_{n \geq 0}$ associated to $(b, 0)$ is called recurrent if

$$\mathbb{P}(X_n = y \text{ infinitely often} \mid X_0 = x) = 1$$

for all $x, y \in V$. It is called transient if for all $x, y \in V$ the above probability is strictly less than 1.

Remark 4.2. Usually one calls a graph $(b, 0)$ recurrent whenever a random walk associated with it is recurrent (see e.g. [Soa1], [Woe]). All the known criteria for recurrence and transience were proven in this context.

Let P be the transition matrix of a random walk associated with $(b, 0)$, i.e. the infinite matrix with entries

$$P(x, y) = \frac{b(x, y)}{\deg(x)}.$$

We can then characterize recurrence in terms of P .

Theorem 5. Let $(X_n)_{n \geq 0}$ be a random walk associated with a connected graph $(b, 0)$. Then $(X_n)_{n \geq 0}$ is recurrent if and only if for all $x, y \in V$

$$\sum_{n=0}^{\infty} P^{(n)}(x, y) = \infty.$$

(here $P^{(n)}$ denotes powers of the matrix P)

Proof. Since $(b, 0)$ is connected, for each $x, y \in V$ there exists an $n \geq 0$ such that $P^{(n)}(x, y) > 0$. Therefore, an analogue computation as in the proof of Proposition 3.2 shows, that the above sum is infinite for all $x, y \in V$ if and only if it is infinite for some $x, y \in V$. Now the statement follows from Proposition 1.17 of [Woe], when taking into account how recurrence is defined there. \square

We will now discuss the relation of recurrence of Dirichlet forms and recurrence of a random walk associated to a weighted graph $(b, 0)$. Assume $m \equiv 1$ and let us write $\tilde{L} = D - A$, where D and A are the two infinite matrices with entries given by

$$D(x, y) = \begin{cases} \deg(x), & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

and

$$A(x, y) = \begin{cases} 0, & \text{if } x = y \\ b(x, y), & \text{if } x \neq y. \end{cases}$$

Then $P = D^{-1}A$ is the transition matrix of a random walk associated to $(b, 0)$ (here D^{-1} is the formal inverse of D , i.e. an infinite matrix having \deg^{-1} on its diagonal).

Theorem 6. *Let $(b, 0)$ be connected and let $e^{-tL^{(D)}}$ be the semigroup associated with $Q^{(D)}$ on $\ell^2(V, 1)$. Then for $x, y \in V$*

$$\int_0^\infty e^{-tL^{(D)}} \delta_x(y) dt = \frac{1}{\deg(x)} \sum_{n=0}^\infty P^{(n)}(x, y).$$

Proof. Let $x, y \in V$. By Proposition 3.3 we obtain

$$\int_0^\infty e^{-tL^{(D)}} \delta_x(y) dt = \lim_{\alpha \rightarrow 0+} (L^{(D)} + \alpha)^{-1} \delta_x(y).$$

Pick an exhaustion (K_i) of V (i.e. $K_i \subseteq K_{i+1}$ and $\cup K_i = V$) such that every K_i is finite and $x \in K_1$. As seen in Theorem A.4, we can compute the resolvent of $Q^{(D)}$ by

$$(L^{(D)} + \alpha)^{-1} \delta_x = \lim_{i \rightarrow \infty} (L_{K_i} + \alpha)^{-1} \delta_x,$$

where L_{K_i} is the restriction of \tilde{L} to $C(K_i)$ (in the sense described there). With A_{K_i}, D_{K_i} we will denote the restrictions of A, D to $C(K_i)$ (in the sense used in Theorem A.4). Let us compute for $y \in K_i$

$$\begin{aligned} (L_{K_i} + \alpha)^{-1} \delta_x(y) &= (I - (D_{K_i} + \alpha)^{-1} A_{K_i})^{-1} (D_{K_i} + \alpha)^{-1} \delta_x(y) \\ &= \frac{1}{\deg(x) + \alpha} (I - (D_{K_i} + \alpha)^{-1} A_{K_i})^{-1} \delta_x(y) \\ &= \frac{1}{\deg(x) + \alpha} \sum_{n=0}^\infty [(D_{K_i} + \alpha)^{-1} A_{K_i}]^n \delta_x(y). \end{aligned} \quad (4.1)$$

Above, we used Neumann series expansion which can be applied since $(D_{K_i} + \alpha)^{-1} A_{K_i}$ is bounded on $\ell^\infty(K_i)$ with norm strictly less than 1. To see this, pick $f \in \ell^\infty(K_i)$ such that $\|f\|_\infty \leq 1$ and observe

$$\begin{aligned} \|(D_{K_i} + \alpha)^{-1} A_{K_i} f\|_\infty &\leq \max_{x \in K_i} \frac{1}{\deg(x) + \alpha} \sum_{y \in K_i} b(x, y) |f(y)| \\ &\leq \max_{x \in K_i} \frac{\deg(x)}{\deg(x) + \alpha} < 1. \end{aligned}$$

To obtain the desired formula we need to pass to the limits under the sum in equation (4.1). For this it suffices to show that convergence in i and afterwards convergence in α is monotone. We show by induction over n that for $y \in K_i$ the inequality

$$[(D_{K_i} + \alpha)^{-1} A_{K_i}]^n \delta_x(y) \leq [(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^n \delta_x(y)$$

holds. The case $n = 0$ is clear. Now assume we have shown the statement for $n - 1$. Using the non-negativity of $[(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^{n-1} \delta_x(z)$ for $z \in K_{i+1}$, we then obtain

$$\begin{aligned} [(D_{K_i} + \alpha)^{-1} A_{K_i}]^n \delta_x(y) &= \frac{1}{\deg(y) + \alpha} \sum_{z \in K_i} b(y, z) [(D_{K_i} + \alpha)^{-1} A_{K_i}]^{n-1} \delta_x(z) \\ (\text{Ind. hypothesis}) &\leq \frac{1}{\deg(y) + \alpha} \sum_{z \in K_i} b(y, z) [(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^{n-1} \delta_x(z) \\ (\text{Nonneg.}) &\leq \frac{1}{\deg(y) + \alpha} \sum_{z \in K_{i+1}} b(y, z) [(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^{n-1} \delta_x(z) \\ &= [(D_{K_{i+1}} + \alpha)^{-1} A_{K_{i+1}}]^n \delta_x(y). \end{aligned}$$

The above implies monotone convergence of the summands in i . A similiar computation shows monotone convergence in α . This finishes the proof. \square

Remark 4.3. • *The theorem above is a version of Theorem 4.34 in [Che]. However, the proof given above is different from the one there.*

- *The proof of the last theorem also provides a precise version of the computation suggested in the discussion previous to Theorem 4.7 in [JP1]. Therefore the theorem above might be considered as an ℓ^2 -analogue to Theorem 4.7 in [JP1].*

Corollary 4.4. *Let $(b, 0)$ be connected. Then a random walk associated with $(b, 0)$ is recurrent if and only if $Q^{(D)}$ as a Dirichlet form on $\ell^2(V, 1)$ is recurrent.*

Proof. This is an immediate consequence of the previous two theorems. \square

Remark 4.5. *Let us briefly discuss the stochastic interpretation of the formula in Theorem 6 by computing the involved quantities from a probabilistic view. We will omit technical details (such as construction of the related processes and measurability issues) and do computations on a formal level.*

At first suppose we are given a Markov process $(X_t)_{t \geq 0}$ in continuous time with values in $V \cup \{\infty\}$ satisfying

$$\mathbb{P}_x(X_t = y) = e^{-tL^{(D)}} \delta_y(x).$$

Here \mathbb{P}_x denotes the probability under the condition that $X_0 = x$ (for a detailed discussion of the relation of Markov processes and regular Dirichlet forms see [FOT]). Let λ be the Lebesgue measure on \mathbb{R} . We then obtain

$$\begin{aligned}\mathbb{E}_x[\lambda\{t > 0 \mid X_t = y\}] &= \int_{\Omega} \int_0^{\infty} 1_{\{t > 0 \mid X_t(\omega) = y\}}(s) ds d\mathbb{P}_x(\omega) \\ &= \int_0^{\infty} \int_{\Omega} 1_{\{t > 0 \mid X_t(\omega) = y\}}(s) d\mathbb{P}_x(\omega) ds \\ &= \int_0^{\infty} \mathbb{P}_x(X_s = y) ds \\ &= \int_0^{\infty} e^{-sL^{(D)}} \delta_y(x) ds.\end{aligned}$$

Therefore, the integral $\int_0^{\infty} e^{-tL^{(D)}} \delta_y(x) dt$ is equal to the expected time that $(X_t)_{t \geq 0}$ spends in y provided that it starts at x .

Now let us assume $(X_n)_{n \geq 0}$ is a random walk associated with $(b, 0)$. Let \sharp denote the counting measure on \mathbb{N} . Then

$$\begin{aligned}\mathbb{E}_x[\sharp\{n \geq 0 \mid X_n = y\}] &= \int_{\Omega} \sum_{n=0}^{\infty} 1_{\{\omega \in \Omega \mid X_n(\omega) = y\}} d\mathbb{P}_x(\omega) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y) \\ &= \sum_{n=0}^{\infty} P^{(n)}(x, y).\end{aligned}$$

This shows that $\sum_{n=0}^{\infty} P^{(n)}(x, y)$ coincides with the expected number of visits of $(X_n)_{n \geq 0}$ to y whenever it started at x .

Chapter 5

Recurrence and Transience

5.1 Classical characterizations of recurrence

Within this section we show continuous time analogues to characterizations of recurrence and transience which are known in the discrete time setting (see [Soa1] and [Woe] for details). We will use the Dirichlet form methods developed above to deduce them and point out where the 'classical' results may be found. At first, we show that $Q^{(D)}$ is transient whenever the graph $(b, 0)$ supports a monopole of finite energy (Theorem 7). Afterwards, we characterize recurrence in terms of properties of the Yamasaki space \mathbf{D} and the capacity of points (Theorem 8). As a last classical characterization of recurrence, we show that it is equivalent to each superharmonic function of finite energy being constant (Theorem 9). The proofs of Theorem 7 and Theorem 9 seem to be new while the one of Theorem 8 was suggested by Daniel Lenz.

A function $u \in \tilde{D}$ will be called a *monopole of finite energy* if there exists some $x \in V$, such that u satisfies

$$\tilde{L}u = \delta_x.$$

The first theorem we want to prove deals with the existence of such monopoles. Its discrete analogue was first shown in [Lyo] (see Theorem 3.33 in [Soa1] also).

Theorem 7. *Let $(b, 0)$ be connected. Then $Q^{(D)}$ is transient if and only if there exists a monopole of finite energy.*

Proof. Assume $u \in \tilde{D}$ satisfies $\tilde{L}u = \delta_w$. We show the transience of $Q^{(D)}$ by constructing a reference function g as in Definition 3.4. Let $v \in C_c(V)$ be given. Then by Lemma 2.4

and Cauchy-Schwarz inequality we obtain

$$|v(w)|m(w) = \langle v, \delta_w \rangle = \langle v, \tilde{L}u \rangle = \tilde{Q}(v, u) \leq Q^{(D)}(v)^{1/2} \tilde{Q}(u)^{1/2}.$$

Since $C_c(V)$ is dense in $D(Q^{(D)})$ with respect to the form norm $\|\cdot\|_Q$, this inequality extends to all $v \in D(Q^{(D)})$. Furthermore, Lemma 2.5 shows that for every $x \in V$ there exists a constant $K_x > 0$ with

$$|v(w) - v(x)| \leq K_x Q^{(D)}(v)^{1/2}$$

for every $v \in D(Q^{(D)})$. Combining these two inequalities we infer the existence of $C_x > 0$ such that for every $v \in D(Q^{(D)})$ the inequality

$$|v(x)| \leq C_x Q^{(D)}(v)^{1/2}$$

holds. Now we define a reference function g by setting

$$g(x) = \frac{a_x}{C_x m(x)},$$

where the a_x are chosen strictly positive such that $g \in \ell^1(V, m) \cap \ell^\infty(V)$ and $\sum a_x = 1$. This finishes the first part of the proof.

On the contrary, let us assume $Q^{(D)}$ is transient. Then by Definition 3.4 there exists a strictly positive $g \in \ell^\infty(V) \cap \ell^1(V, m)$ such that

$$\langle |u|, g \rangle \leq Q^{(D)}(u)^{1/2}, \text{ for every } u \in D(Q^{(D)}).$$

By the definition of $D(Q^{(D)})_e$ and Lemma 3.7, the above inequality extends to all $u \in D(Q^{(D)})_e$. For fixed $w \in V$, this implies the continuity of the linear functional

$$F_w : D(Q^{(D)})_e \rightarrow \mathbb{R}, u \mapsto u(w),$$

with respect to the inner product $Q^{(D)}$. Theorem 3 yields that $(D(Q^{(D)})_e, Q^{(D)})$ is a Hilbert space. Thus by Riesz representation theorem there exists a function $v \in D(Q^{(D)})_e$ such that

$$u(w) = F_w(u) = Q^{(D)}(u, v)$$

for all $u \in D(Q^{(D)})_e$. By Lemma 3.7 we already know $D(Q^{(D)})_e \subseteq \tilde{D}$. Applying Lemma 2.4 we can compute

$$(\tilde{L}v)(x) = \frac{1}{m(x)} \langle \tilde{L}v, \delta_x \rangle = \frac{1}{m(x)} \tilde{Q}(v, \delta_x) = \frac{1}{m(x)} Q^{(D)}(v, \delta_x) = \frac{1}{m(x)} \delta_x(w).$$

This shows $\tilde{L}(m(w)v) = \delta_w$ and therefore finishes the proof. \square

As a next step, we want to prove an analogue to Theorem 3.63 in [Soa1]. It deals with the structure of the space \mathbf{D} and shows that recurrence is equivalent to points having capacity zero, where the *capacity* of $x \in V$ is defined by

$$\text{cap}(x) = \inf\{Q^{(D)}(v) : v \in C_c(V), v(x) = 1\}.$$

Theorem 8. *Let $(b, 0)$ be connected. The following assertions are equivalent:*

- (i) $Q^{(D)}$ is recurrent.
- (ii) $C_c(V)$ is dense in \mathbf{D} .
- (iii) The constant function 1 can be approximated in \mathbf{D} by function of $C_c(V)$. In this case the approximating functions e_n can be chosen to satisfy $0 \leq e_n \leq 1$.
- (iv) $\text{cap}(o) = \inf\{Q^{(D)}(v) : v \in C_c(V), v(o) = 1\} = 0$.

Proof. '(i) \Rightarrow (iii)': By the recurrence of $Q^{(D)}$, Theorem 4 yields the existence of a sequence $(f_n) \subseteq D(Q^{(D)})$, such that $f_n \rightarrow 1$ with respect to $\|\cdot\|_o$. Furthermore, $C_c(V)$ is dense in $D(Q^{(D)})$ with respect to $\|\cdot\|_Q$. Thus, there exist $\tilde{e}_n \in C_c(V)$ such that

$$\|\tilde{e}_n - f_n\|_Q \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $e_n = (0 \vee \tilde{e}_n) \wedge 1$. We then obtain by the Markov property of $Q^{(D)}$

$$Q^{(D)}(e_n)^{1/2} \leq Q^{(D)}(\tilde{e}_n)^{1/2} \leq Q^{(D)}(\tilde{e}_n - f_n)^{1/2} + Q^{(D)}(f_n)^{1/2}.$$

Because $c \equiv 0$, the right side of the above inequality needs to converge to zero by construction. It is straightforward that $e_n \rightarrow 1$ pointwise. This implies

$$\|1 - e_n\|_o \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which shows (iii).

'(iii) \Rightarrow (ii)': This proof will be done in two steps.

Step 1: Let $u \in \mathbf{D}$, such that $0 \leq u \leq 1$ and let $(e_n) \subseteq C_c(V)$ be a sequence approximating 1 in \mathbf{D} , such that $0 \leq e_n \leq 1$. Then by Corollary 2.8 the sequence $u \wedge e_n$ converges to u with respect to $\|\cdot\|_o$. Furthermore $u \wedge e_n \in C_c(V)$ showing that

$$u \in \overline{C_c(V)}^{\|\cdot\|_o}.$$

Step 2: Let $u \in \mathbf{D}$ such that $u \geq 0$. Then Corollary 2.7 yields the convergence of $u \wedge N$ to u with respect to $\|\cdot\|_o$ as $N \rightarrow \infty$. Step 1 allows us to approximate $u \wedge N$ by functions

of $C_c(V)$. For general $u \in \mathbf{D}$ we can split u in its positive and negative part which both belong to \mathbf{D} . This shows (ii).

'(ii) \Rightarrow (i)': Proposition 3.8 shows that $D(Q^{(D)})_e$ is the closure of $D(Q^{(D)})$ in \mathbf{D} . Since $C_c(V) \subseteq D(Q^{(D)})$, condition (ii) implies $D(Q^{(D)})_e = \mathbf{D}$. From this and $c \equiv 0$ we infer $1 \in D(Q^{(D)})_e$ and $Q^{(D)}(1) = 0$. Now Theorem 4 yields (i).

'(iii) \Rightarrow (iv)': This is obvious noting that the sequence in (iii) can be chosen to satisfy $e_n(o) = 1$.

'(iv) \Rightarrow (i)': Assume $Q^{(D)}$ is transient. Then by Definition 3.4 there exists a constant $C > 0$, such that for any $v \in D(Q^{(D)})$ the inequality $Q^{(D)}(v)^{1/2} \geq C|v(o)|$ holds. In particular

$$\inf\{Q^{(D)}(v) : v \in C_c(V), v(o) = 1\} \geq C^2 > 0.$$

□

The last classical result that we want to prove deals with *superharmonic functions of finite energy*, i.e. functions $u \in \tilde{D}$ satisfying

$$\tilde{L}u \geq 0.$$

It is an analogue to Theorem 3.34 of [Soa1].

Theorem 9. *Let $(b, 0)$ be connected. Then $Q^{(D)}$ is recurrent if and only if any superharmonic function of finite energy is constant.*

Proof. Assume $Q^{(D)}$ is recurrent and let $u \in \tilde{D}$, such that $\tilde{L}u \geq 0$. As a first step we show that $\tilde{L}u = 0$ must hold. Assume there exists a $w \in V$, such that $\tilde{L}u(w) > 0$. Then by Lemma 2.4 we obtain for all $v \in C_c(V)$

$$|v(w)|(\tilde{L}u)(w)m(w) \leq \langle |v|, \tilde{L}u \rangle = \tilde{Q}(|v|, u) \leq Q^{(D)}(v)^{1/2} \tilde{Q}(u)^{1/2}.$$

Following the first part of the proof of Theorem 7, such an inequality implies transience of $Q^{(D)}$. Hence we conclude $\tilde{L}u = 0$.

Secondly, let us show $|u(x) - u(y)| = 0$ for all $x, y \in V$. Since $Q^{(D)}$ is recurrent part (ii) of Theorem 8 implies the existence of a sequence $(u_n) \subseteq C_c(V)$, such that $\|u - u_n\|_o \rightarrow 0$. Furthermore, for each $x, y \in V$, Lemma 2.5 provides constants $K_{x,y} > 0$, such that

$$|u(x) - u(y)| \leq K_{x,y} \tilde{Q}(u)^{1/2}$$

holds. From the above and Lemma 2.4 we conclude

$$|u(x) - u(y)| \leq K_{x,y} \tilde{Q}(u)^{1/2} = K_{x,y} \lim_{n \rightarrow \infty} \tilde{Q}(u, u_n)^{1/2} = K_{x,y} \lim_{n \rightarrow \infty} \langle \tilde{L}u, u_n \rangle^{1/2} = 0.$$

This proves one implication.

On the contrary assume $Q^{(D)}$ is transient. Then by Theorem 7 there exists a monopole of finite energy. This monopole clearly is superharmonic and nonconstant. \square

Remark 5.1. • *In the literature the normalized Laplacian $\tilde{L}_{b,0,\deg}$ is used to state analogues to Theorem 7 and Theorem 9 for the discrete time case.*

- *In the case where $m \equiv 1$ we could have deduced the results of this section (with the normalized Laplacian instead of \tilde{L}) from Corollary 4.4 and the corresponding results found in the literature.*
- *Let us stress that all the above characterizations of recurrence are measure independent, i.e. if they are satisfied for one measure they hold for all (see Theorem 17 for more details). This shows that recurrence of $Q^{(D)}$ only depends on the structure of (b, c) and not on the underlying ℓ^2 -space. Therefore, in Corollary 4.4, the restriction to the case $m \equiv 1$ may be dropped. This means that $Q^{(D)}$ as a Dirichlet form on $\ell^2(V, m)$ is recurrent if and only if the random walk associated with $(b, 0)$ is recurrent.*

5.2 New criteria for recurrence

Within this section we want to provide two more criteria for recurrence which seem to be new. The first one asks whether a certain integral vanishes (Theorem 10), while the second one deals with the validity of Green's formula for a different situation than in Lemma 2.4 (see Theorem 11). Both criteria were motivated by recent works. The first one is an analogue to a result of [GM], while the second one is a version of Theorem 4.6 in [JP1] for not necessarily locally finite graphs.

Theorem 10. *Let $(b, 0)$ be connected. Then $Q^{(D)}$ is recurrent if and only if*

$$\sum_{x \in V} \tilde{L}u(x)m(x) = 0$$

for all $u \in \tilde{D}$, such that $\tilde{L}u \in \ell^1(V, m)$.

Proof. Let $Q^{(D)}$ be recurrent and $u \in \tilde{D}$, such that $\tilde{L}u \in \ell^1(V, m)$. By Theorem 8 there exists a sequence e_n in $C_c(V)$ satisfying $\|e_n - 1\|_o \rightarrow 0$ and $0 \leq e_n \leq 1$. We infer by Lebesgue's theorem and Lemma 2.4

$$\begin{aligned} \sum_{x \in V} \tilde{L}u(x)m(x) &= \lim_{n \rightarrow \infty} \sum_{x \in V} e_n(x) \tilde{L}u(x)m(x) \\ &= \lim_{n \rightarrow \infty} \tilde{Q}(e_n, u) = 0. \end{aligned}$$

On the contrary assume $Q^{(D)}$ is transient. Then Theorem 7 yields the existence of a function $v \in \tilde{D}$ satisfying $\tilde{L}v = \delta_w \in \ell^1(V, m)$ for some $w \in V$. Obviously

$$\sum_{x \in V} \tilde{L}v(x)m(x) = m(w) \neq 0.$$

This finishes the proof. □

For proving Green's formula in Lemma 2.4 we needed that one of the functions had compact support. As a last characterization of recurrence in this section we want to show that recurrence is equivalent to the validity of Green's formula for a different class of functions. To do this, let us introduce the *boundary term*

$$R : D_\infty \times D^1 \rightarrow \mathbb{R}$$

given by

$$R(u, v) = \tilde{Q}(u, v) - \langle u, \tilde{L}v \rangle.$$

Here we used the notation $D_\infty = \tilde{D} \cap \ell^\infty(V)$ and $D^1 = \{v \in \tilde{D} \mid \tilde{L}v \in \ell^1(V, m)\}$. Now the following theorem holds.

Theorem 11. *Let $(b, 0)$ be connected. Then $Q^{(D)}$ is recurrent if and only if $R \equiv 0$.*

Proof. Assume $Q^{(D)}$ is recurrent. Let $u \in D_\infty$ and $v \in D^1$ be given. Then Theorem 8 yields the existence of a sequence $(u_n) \subseteq C_c(V)$ converging to u with respect to $\|\cdot\|_o$. Without loss of generality, this sequence can be chosen to be uniformly bounded by $\|u\|_\infty$. Then Lebesgue's theorem and Lemma 2.4 yield

$$\tilde{Q}(u, v) = \lim_{n \rightarrow \infty} \tilde{Q}(u_n, v) = \lim_{n \rightarrow \infty} \langle u_n, \tilde{L}v \rangle = \langle u, \tilde{L}v \rangle.$$

This implies $R \equiv 0$.

On the contrary assume $Q^{(D)}$ is transient. By Theorem 7 there exists a function $v \in D^1$, such that

$$\sum_{x \in V} \tilde{L}v(x)m(x) \neq 0.$$

Since $1 \in D_\infty$ and $\tilde{Q}(1, v) = 0$, this implies $R(1, v) \neq 0$ which finishes the proof. \square

Remark 5.2. *The above theorem was motivated by Theorem 4.6 of [JP1]. This theorem deals with a boundary term pairing a space spanned by certain monopoles and dipoles and the functions of finite energy. However, the boundary representation that is used to deduce the result there seems to hold true only for locally finite graphs. We will explain some details of their computation below.*

Let us now consider the case where $(b, 0)$ is locally finite. Then we can compute the boundary term R by a limiting procedure. First we fix some notation. For a subgraph $W \subseteq V$ let

$$\text{bd}W = \{x \in W \mid \text{there exists } y \in V \setminus W \text{ such that } x \sim y\}$$

be the set of all vertices in W , which are connected with the complement of W . Furthermore, let

$$\text{int}W = W \setminus \text{bd}W.$$

Note that $x \in \text{int}W$ and $y \sim x$ implies $y \in W$. For $u \in \tilde{D}$ and $x \in \text{bd}W$ we introduce the outward normal derivative with respect to W , given by

$$(\partial_W u)(x) = \sum_{y \in W} b(x, y)(u(x) - u(y)).$$

Then the following holds.

Proposition 5.3. *Let $(b, 0)$ be locally finite. Then for $u \in D_\infty$ and $v \in D^1$ the boundary term R may be computed via*

$$R(u, v) = \lim_{n \rightarrow \infty} \sum_{x \in \text{bd}V_n} u(x)(\partial_{V_n} v)(x).$$

Here (V_n) is any increasing sequence of finite subsets of V , such that $\cup_n V_n = V$.

Proof. Let V_n be as above. Then a simple calculation shows

$$\frac{1}{2} \sum_{x, y \in V_n} b(x, y)(u(x) - u(y))(v(x) - v(y)) = \sum_{x \in \text{int}V_n} u(x)(\tilde{L}v)(x)m(x) + \sum_{x \in \text{bd}V_n} u(x)(\partial_{V_n} v)(x).$$

Because $(b, 0)$ is locally finite, we obtain $\cup_n \text{int}V_n = V$. Furthermore, our assumptions yield that the sum on the left and

$$\sum_{x \in V} u(x)(\tilde{L}v)(x)m(x)$$

are absolutely convergent. Taking the limit $n \rightarrow \infty$ implies the desired statement, noting that absolute convergence yields independence of the choice of the V_n . \square

Remark 5.4. *The local finiteness is crucial for the above computations, which are taken from the proof of Theorem 4.6 in [JP1]. Otherwise one cannot control $\text{int } W$ for finite sets W . In the non local finite case it might even happen that $\text{int } W = \emptyset$ for all finite $W \subseteq V$.*

Chapter 6

Further global properties

In this chapter we want to discuss two other concepts - namely stochastic completeness and the validity of $Q^{(D)} = Q^{(N)}$. It turns out that characterizations of these two properties are similar to the ones obtained for recurrence and transience. We first introduce the notion of stochastic completeness (Definition 6.1) and then prove a characterization analogue to Theorem 10 (see Theorem 12) which is also motivated by results of [GM]. Afterwards we present a criterion for stochastic completeness in terms of the unique solvability of the equation $(\tilde{L} + \alpha)u = 0$ on ℓ^∞ (Theorem 13). This criterion is taken from [KL]. We then characterize when the Neumann form $Q^{(N)}$ and the regular Dirichlet form $Q^{(D)}$ coincide. We show that this is related to unique solvability of $(\tilde{L} + \alpha)u = 0$ on $\tilde{D} \cap \ell^2(V, m)$ and the validity of Green's formula for ℓ^2 -functions (Theorem 14). The connection between $Q^{(D)} = Q^{(N)}$ and the validity of $\tilde{Q}(u, v) = \langle \tilde{L}u, v \rangle$ for certain ℓ^2 -functions seems to be new.

6.1 Stochastic completeness

Using the extension of a Markovian resolvent to $\ell^\infty(V)$ (see appendix for details) we can introduce the concept of stochastic completeness.

Definition 6.1. *Let Q be a Dirichlet form associated with (b, c) . Q is called stochastically complete if*

$$(L + 1)^{-1}1 = 1.$$

Otherwise Q is called stochastically incomplete.

Remark 6.2. *By some general principles (the correspondence $(L + \alpha)^{-1} \leftrightarrow e^{-tL}$) the above definition of stochastic completeness is equivalent to the validity of*

$$e^{-tL}1 = 1$$

for all $t > 0$. This equation is important whenever one investigates a Markov process $(X_t)_{t \geq 0}$ on $V \cup \{\infty\}$, which satisfies

$$\mathbb{P}(X_t = y | X_0 = x) = e^{-tL} \delta_y(x).$$

In view of that, stochastic completeness is equivalent to

$$\mathbb{P}(X_t \in V | X_0 = x) = 1 \text{ for all } t > 0.$$

In other words, stochastic completeness describes the property that X_t does not leave V in finite time.

The next result is similar to Theorem 10. It seems to be new in this context.

Theorem 12. *Let (b, c) be connected. $Q^{(D)}$ is stochastically complete if and only if*

$$\sum_{x \in V} (\tilde{L}u)(x)m(x) = 0$$

for all $u \in D(Q^{(D)}) \cap \ell^1(V, m)$, such that $\tilde{L}u \in \ell^1(V, m) \cap \ell^2(V, m)$.

Proof. Pick a sequence (e_n) in $C_c(V)$, which satisfies $0 \leq e_n \leq e_{n+1} \leq 1$ such that $e_n \rightarrow 1$ pointwise. Let $u_n = (L^{(D)} + 1)^{-1}e_n \in D(L^{(D)})$. The way we extended the resolvent to ℓ^∞ yields pointwise convergence of (u_n) towards $(L^{(D)} + 1)^{-1}1$. Furthermore, because $(L^{(D)} + 1)^{-1}$ is a positivity preserving, Markovian resolvent, we infer $0 \leq u_n \leq 1$.

Now assume $Q^{(D)}$ is stochastically complete and let $u \in D(Q^{(D)}) \cap \ell^1(V, m)$, such that $\tilde{L}u \in \ell^1(V, m) \cap \ell^2(V, m)$ be given. Proposition 2.10 shows that u belongs to $D(L^{(D)})$. Furthermore, stochastic completeness yields pointwise convergence of (u_n) towards 1. Using the self-adjointness of $L^{(D)}$ and Lebesgue's theorem we may compute

$$\begin{aligned} \sum_{x \in V} (\tilde{L}u)(x)m(x) &= \sum_{x \in V} (L^{(D)}u)(x)m(x) \\ &= \lim_{n \rightarrow \infty} \sum_{x \in V} u_n(x)(L^{(D)}u)(x)m(x) \\ &= \lim_{n \rightarrow \infty} \sum_{x \in V} (L^{(D)}u_n)(x)u(x)m(x) \\ &= \lim_{n \rightarrow \infty} \sum_{x \in V} (e_n(x) - u_n(x))u(x)m(x) \\ &= \sum_{x \in V} (1 - (L^{(D)} + 1)^{-1}1(x))u(x)m(x) \\ &= 0. \end{aligned}$$

This shows one implication.

On the contrary, if the sum is always vanishing, put $u = (L^{(D)} + 1)^{-1}v$, where $v \in \ell^1(V, m) \cap \ell^2(V, m)$ is chosen strictly positive. This implies that u belongs to $D(Q^{(D)}) \cap \ell^1(V, m)$ (see appendix, extension of the resolvent to ℓ^1) and $\tilde{L}u \in \ell^1(V, m) \cap \ell^2(V, m)$. Then our assumptions, Lebesgue's theorem and the self-adjointness of the resolvent yield

$$\begin{aligned}
 0 &= \sum_{x \in V} (\tilde{L}u)(x)m(x) \\
 &= \lim_{n \rightarrow \infty} \sum_{x \in V} e_n(x)(L^{(D)}u)(x)m(x) \\
 &= \lim_{n \rightarrow \infty} \sum_{x \in V} e_n(x)(v(x) - (L^{(D)} + 1)^{-1}v(x))m(x) \\
 &= \lim_{n \rightarrow \infty} \sum_{x \in V} (e_n(x) - (L^{(D)} + 1)^{-1}e_n(x))v(x)m(x) \\
 &= \sum_{x \in V} (1 - (L^{(D)} + 1)^{-1}1(x))v(x)m(x).
 \end{aligned}$$

Since v was chosen strictly positive and $(L^{(D)} + 1)^{-1}1(x) \leq 1$ (see appendix) this shows $(L^{(D)} + 1)^{-1}1 = 1$. \square

Remark 6.3. *As for recurrence one can show that on a connected graph a nonvanishing potential c implies stochastic incompleteness (see e.g. [KL]). Therefore, we will assume $c \equiv 0$ in the following sections.*

The next theorem is a characterization of stochastic completeness in terms of the unique solvability of $(\tilde{L} + \alpha)u = 0$ on $\ell^\infty(V)$. We will need it for the discussion in the next chapter.

Theorem 13. *Let $Q^{(D)}$ be the regular Dirichlet form associated with $(b, 0)$. Then the following assertions are equivalent:*

(i) $Q^{(D)}$ is stochastically complete.

(ii) For any $\alpha > 0$ the equation $(\tilde{L} + \alpha)u = 0$ is uniquely solvable on $\ell^\infty(V)$.

Proof. This is an immediate consequence of Theorem 1 in [KL]. \square

6.2 $Q^{(D)} = Q^{(N)}$

From the definition of $Q^{(D)}$ and $Q^{(N)}$ it is not clear whether these two forms coincide. The theorem below provides a characterization of this in terms of unique solvability of $(\tilde{L} + \alpha)u = 0$ on $\tilde{D} \cap \ell^2(V, m)$ and the validity of Green's formula for ℓ^2 -functions.

Theorem 14. *The following assertions are equivalent:*

- (i) $Q^{(D)} = Q^{(N)}$.
- (ii) For any $\alpha > 0$ the equation $(\tilde{L} + \alpha)u = 0$ is uniquely solvable in $\tilde{D} \cap \ell^2(V, m)$.
- (iii) For all $u \in \tilde{D} \cap \ell^2(V, m)$ and $v \in \tilde{D} \cap \ell^2(V, m)$, such that $\tilde{L}v \in \ell^2(V, m)$

$$\tilde{Q}(u, v) = \langle u, \tilde{L}v \rangle$$

holds.

Proof. $'(ii) \Rightarrow (i)'$: By general theory (see appendix) it suffices to show that the resolvents $(L^{(D)} + \alpha)^{-1}$ and $(L^{(N)} + \alpha)^{-1}$ coincide. Let $u \in \ell^2(V, m)$ be arbitrary. Set

$$v = (L^{(N)} + \alpha)^{-1}u - (L^{(D)} + \alpha)^{-1}u.$$

Because both operators $L^{(D)}$ and $L^{(N)}$ are restrictions of \tilde{L} to their corresponding domains, we infer

$$(\tilde{L} + \alpha)v = 0.$$

Therefore (ii) implies $v = 0$ as was to be shown.

$'(i) \Rightarrow (ii)'$: As seen in Proposition 2.10, the domain of $L^{(D)}$ is given by

$$D(L^{(D)}) = \{u \in D(Q^{(D)}) \mid \tilde{L}u \in \ell^2(V, m)\}.$$

Because $Q^{(D)} = Q^{(N)}$ we conclude

$$D(L^{(D)}) = \{u \in \tilde{D} \cap \ell^2(V, m) \mid \tilde{L}u \in \ell^2(V, m)\}.$$

Now let $\alpha > 0$ and $u \in \tilde{D} \cap \ell^2(V, m)$ be given, such that $\tilde{L}u = -\alpha u$. By the above this implies $u \in D(L^{(D)})$. Because $L^{(D)}$ is positive, i.e. its spectrum is contained in $[0, \infty)$, we infer $u = 0$.

$'(i) \Rightarrow (iii)'$: Assume $Q^{(D)} = Q^{(N)}$. Then Proposition 2.10 implies that

$$D(L^{(N)}) = D(L^{(D)}) = \{v \in D(Q^{(D)}) \mid \tilde{L}v \in \ell^2(V, m)\} = \{v \in \tilde{D} \cap \ell^2(V, m) \mid \tilde{L}v \in \ell^2(V, m)\}.$$

This shows (iii).

$'(iii) \Rightarrow (i)'$: Assume $\tilde{Q}(u, v) = \langle u, \tilde{L}v \rangle$ for all $u \in \tilde{D} \cap \ell^2(V, m)$ and $v \in \tilde{D} \cap \ell^2(V, m)$ such that $\tilde{L}v \in \ell^2(V, m)$. Then by the correspondence $L^{(N)} \leftrightarrow Q^{(N)}$ the domain of $L^{(N)}$ satisfies

$$D(L^{(N)}) \supseteq \{v \in \tilde{D} \cap \ell^2(V, m) \mid \tilde{L}v \in \ell^2(V, m)\}.$$

Now Proposition 2.10 shows $L^{(D)} \subseteq L^{(N)}$. Taking adjoints yields the statement. \square

Remark 6.4. *The equivalence (i) \Leftrightarrow (ii) was already shown in Corollary 3.3 of [HKLW]. However their proof is different from the one given above. (i) \Leftrightarrow (iii) seems to be new.*

Part (iii) of the theorem above can be considered as a boundary term characterization of $Q^{(D)} = Q^{(N)}$ which is an analogue to Theorem 11. To see this let us introduce the boundary term

$$\hat{R} : \tilde{D} \cap \ell^2(V, m) \times \{u \in \tilde{D} \cap \ell^2(V, m) \mid \tilde{L}u \in \ell^2(V, m)\} \rightarrow \mathbb{R}$$

acting as

$$\hat{R}(u, v) = \tilde{Q}(u, v) - \langle u, \tilde{L}u \rangle.$$

Then the following holds.

Corollary 6.5. *$Q^{(D)} = Q^{(N)}$ if and only if $\hat{R} \equiv 0$.*

Chapter 7

Consequences of recurrence

Within this chapter we want to discuss the relationship between all the global properties discussed above. We prove that recurrence of $Q^{(D)}$ always implies stochastic completeness and $Q^{(D)} = Q^{(N)}$ (see Theorem 15) and that all concepts coincide in the case where m is finite (Theorem 16). Using these results we finally show that recurrence of $Q^{(D)}$ is related to the unique solvability of the eigenvalue equation $(\tilde{L} + \alpha)u = 0$ in \tilde{D} . (Theorem 17). Except Theorem 15, which is valid in much more general situations, all the results of this chapter seem to be new in this context.

The next two lemmas are used to prove that recurrence implies $Q^{(D)} = Q^{(N)}$.

Lemma 7.1. *Assume $(b, 0)$ is connected, $Q^{(D)}$ is recurrent and let $\alpha > 0$. Let $u \in \tilde{D}$, such that $u \leq 0$ and $(\tilde{L} + \alpha)u \geq 0$. Then $u \equiv 0$.*

Proof. Let u be as above. By $\tilde{L}u \geq -\alpha u$ we infer that u is superharmonic. Theorem 9 implies that u is constant. We then obtain

$$0 \leq (\tilde{L} + \alpha)u = \alpha u \leq 0.$$

This shows $u \equiv 0$ and finishes the proof. □

From this we can deduce the following uniqueness statement for solutions to the equation $(\tilde{L} + \alpha)u = 0$ in \tilde{D} .

Lemma 7.2. *Assume $(b, 0)$ is connected, $Q^{(D)}$ is recurrent and $\alpha > 0$. Let $u \in \tilde{D}$, such that $(\tilde{L} + \alpha)u = 0$. Then $u \equiv 0$.*

Proof. Let $u \in \tilde{D}$ be given, such that $(\tilde{L} + \alpha)u = 0$. With $u_+ = u \wedge 0$ and $u_- = (-u) \wedge 0$ we denote the positive/negative part of u . Because $u_+, u_- \in \tilde{D}$ it suffices to show

$(\tilde{L} + \alpha)u_+ \leq 0$ and $(\tilde{L} + \alpha)u_- \leq 0$ to obtain the statement by the lemma above. The assumption $(\tilde{L} + \alpha)u = 0$ implies

$$(\tilde{L} + \alpha)u_+ = (\tilde{L} + \alpha)u_-,$$

which is equivalent to

$$(\tilde{L} + \alpha)u_+(x) = \frac{\deg(x)}{m(x)}u_-(x) - \frac{1}{m(x)} \sum_{y \in V} b(x, y)u_-(y) + \alpha u_-(x).$$

Thus if $u(x) \geq 0$, we obtain

$$(\tilde{L} + \alpha)u_+(x) = -\frac{1}{m(x)} \sum_{y \in V} b(x, y)u_-(y) \leq 0.$$

Furthermore if $u(x) < 0$, we conclude by the definition of \tilde{L}

$$(\tilde{L} + \alpha)u_+(x) = -\frac{1}{m(x)} \sum_{y \in V} b(x, y)u_+(y) \leq 0.$$

This finishes the proof. \square

We are now able to prove that recurrence implies stochastic completeness and $Q^{(D)} = Q^{(N)}$.

Theorem 15. *Let $(b, 0)$ be connected and $Q^{(D)}$ be recurrent. Then $Q^{(D)}$ is stochastically complete and $Q^{(D)} = Q^{(N)}$.*

Proof. Stochastic completeness: This is an immediate consequence of Theorem 10 and Theorem 12.

$Q^{(D)} = Q^{(N)}$: This is an immediate consequence of Lemma 7.2 and Theorem 14. \square

We can now show that in the case where m is finite all of the above concepts coincide.

Theorem 16. *Let $(b, 0)$ be connected and $m(V) < \infty$. Then the following assertions are equivalent:*

- (i) $Q^{(D)}$ is recurrent.
- (ii) $Q^{(D)}$ is stochastically complete.
- (iii) $Q^{(D)} = Q^{(N)}$.

Proof. '(i) \Rightarrow (ii)': This is one implication of Theorem 15.

'(iii) \Rightarrow (i)': The finiteness of m implies $1 \in D(Q^{(N)}) = D(Q^{(D)})$. Since $c \equiv 0$, we conclude $Q^{(D)}(1) = 0$. Now recurrence of $Q^{(D)}$ follows from Theorem 4.

'(ii) \Rightarrow (iii)': Let us assume $Q^{(D)} \neq Q^{(N)}$. Then the resolvents $(L^{(N)} + 1)^{-1}$ and $(L^{(D)} + 1)^{-1}$ must be different. Because $\ell^\infty(V) \cap \ell^2(V, m)$ is dense in $\ell^2(V, m)$, there exists a bounded u , such that

$$(L^{(D)} + 1)^{-1}u \neq (L^{(N)} + 1)^{-1}u.$$

Both functions $(L^{(D)} + 1)^{-1}u$ and $(L^{(N)} + 1)^{-1}u$ are bounded solutions to the equation

$$(\tilde{L} + 1)v = u.$$

Thus $(\tilde{L} + 1)$ is not injective on $\ell^\infty(V)$. This implies stochastic incompleteness by Theorem 13. \square

We have already seen that stochastic completeness and the equality $Q^{(D)} = Q^{(N)}$ are related to the uniqueness of solutions of $(\tilde{L} + \alpha)v = 0$ on certain spaces. Using the last theorem, we want to prove an analogue statement for recurrence. However, as we pointed out before, recurrence does not depend on the choice of m . Thus the uniqueness statement, which is equivalent to recurrence, must be stronger than the ones for the other concepts. Let us write $Q_m^{(D)}$ whenever we refer to $Q^{(D)}$ on $\ell^2(V, m)$ and let \tilde{L}_m be the corresponding formal operator. Furthermore, by $\tilde{\Delta}$ we denote an operator on \tilde{D} acting as

$$(\tilde{\Delta}u)(x) := (\tilde{L}_{b,0,1}u)(x) = \sum_{y \in V} b(x, y)(u(x) - u(y)).$$

Theorem 17. *Let $(b, 0)$ be connected. The following assertions are equivalent:*

- (i) *For some measure of full support m on V the form $Q_m^{(D)}$ is recurrent.*
- (ii) *For all measures of full support m on V the form $Q_m^{(D)}$ is recurrent.*
- (iii) *For all measures of full support m and for any $\alpha > 0$ the equation $(\tilde{L}_m + \alpha)u = 0$ has a unique solution in \tilde{D} .*
- (iv) *For some finite measure of full support m and for any $\alpha > 0$ the equation $(\tilde{L}_m + \alpha)u = 0$ has a unique solution in \tilde{D} .*
- (v) *For all $v : V \rightarrow (0, \infty)$ the equation $(\tilde{\Delta} + v)u = 0$ has a unique solution in \tilde{D} .*

(vi) For some $v : V \rightarrow (0, \infty)$ which belongs to $\ell^1(V, 1)$ the equation $(\tilde{\Delta} + v)u = 0$ has a unique solution in \tilde{D} .

Proof. $'(i) \Rightarrow (ii)'$: It suffices to show the statement for transience. Let m be a measure, such that $Q_m^{(D)}$ is transient and let m' be another measure of full support. Theorem 7 shows that there exists a function $v \in \tilde{D}$ and $w \in V$, such that

$$\tilde{L}_m v = \delta_w.$$

Then

$$\tilde{L}_{m'} v = \frac{m(w)}{m'(w)} \delta_w.$$

This implies the existence of a monopole with respect to $\tilde{L}_{m'}$ and Theorem 7 shows transience of $Q_{m'}^{(D)}$.

$'(ii) \Rightarrow (iii)'$: This follows from Lemma 7.2.

$'(iii) \Rightarrow (iv)'$: This is obvious.

$'(iv) \Rightarrow (i)'$: By the uniqueness of the solution, we infer $Q_m^{(D)} = Q_m^{(N)}$ from Theorem 14. Since m is finite, Theorem 16 implies recurrence of $Q_m^{(D)}$.

$'(v) \Leftrightarrow (iii)'$: This is clear since any $v : V \rightarrow (0, \infty)$ can be written in the form $v = \alpha m$ and vice versa. Then $(\tilde{\Delta} + v)u = 0$ if and only if $(\tilde{L}_m + \alpha)u = 0$.

$'(v) \Rightarrow (vi)'$: This is clear.

$'(vi) \Rightarrow (i)'$: Let $v : V \rightarrow (0, \infty)$, which belongs to $\ell^1(V, 1)$, such that

$$(\tilde{\Delta} + v)u = 0$$

has a unique solution in \tilde{D} . Then the above is obviously equivalent to

$$(\tilde{L}_v + 1)u = 0$$

being uniquely solvable in \tilde{D} . We infer that $(L^{(D)} + 1)^{-1}$ and $(L^{(N)} + 1)^{-1}$ must agree (as resolvents associated with $Q^{(D)}, Q^{(N)}$ on $\ell^2(V, v)$). This shows $L^{(D)} = L^{(N)}$ which implies $Q^{(D)} = Q^{(N)}$. Since v is a finite measure on V , we can conclude by Theorem 16 that $Q_v^{(D)}$ is recurrent arriving at (i) . \square

Appendix A

General results

In this appendix we provide known results, which would not fit in the main text. The first part is devoted to general theory of Dirichlet forms, while the second one deals with some results about Dirichlet forms on graphs. The last part provides two theorems about vector valued integrals.

A.1 Dirichlet forms and associated objects

Definition A.1. Let $D(Q)$ be a dense subspace of $\ell^2(V, m)$. A map

$$Q : D(Q) \times D(Q) \rightarrow \mathbb{R}$$

is called *Dirichlet form* if the following conditions are satisfied:

(Q1) $Q(f, f) \geq 0$, $Q(f, g) = Q(g, f)$ and $Q(\alpha f + g, h) = \alpha Q(f, h) + Q(g, h)$ for all $f, g, h \in D(Q)$, $\alpha \in \mathbb{R}$. (Linearity)

(Q2) $D(Q)$ equipped with the inner product

$$\langle f, g \rangle_Q = Q(f, g) + \langle f, g \rangle$$

is a Hilbert space.

(Closedness)

(Q3) For any normal contraction C (i.e. a function $C : \mathbb{R} \rightarrow \mathbb{R}$ with $C(0) = 0$ and $|C(x) - C(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$) and any $u \in D(Q)$ the function $C \circ u$ belongs to $D(Q)$ and the inequality

$$Q(C \circ u) \leq Q(u)$$

holds.

(Markov property)

We will call a Dirichlet form *regular* if $C_c(V)$ is contained in $D(Q)$ and

$$\overline{C_c(V)}^{\|\cdot\|_Q} = D(Q),$$

where $\|\cdot\|_Q$ is the norm given by

$$\|\cdot\|_Q = \sqrt{\langle \cdot, \cdot \rangle_Q}.$$

Definition A.2. A family $(T_t)_{t>0}$ of bounded linear operators on $\ell^2(V, m)$ is called strongly continuous Markovian semigroup if the following conditions are satisfied:

- (S1) For any $t > 0$ the operator T_t is self-adjoint. (Symmetry)
- (S2) $T_{t+s} = T_t T_s$ for every $t, s > 0$. (Semigroup property)
- (S3) $\|T_t f\|_2 \leq \|f\|_2$ for every $t > 0, f \in \ell^2(V, m)$. (Contractivity)
- (S4) $\|T_t f - f\|_2 \rightarrow 0$ as $t \rightarrow 0$ for every $f \in \ell^2(V, m)$. (Strong continuity)
- (S5) $0 \leq T_t f \leq 1$ for $f \in \ell^2(V, m)$ with $0 \leq f \leq 1$. (Markov property)

A family $(G_\alpha)_{\alpha>0}$ of bounded linear operators on $\ell^2(V, m)$ is called strongly continuous Markovian resolvent if the following conditions are satisfied:

- (R1) For any $\alpha > 0$ the operator G_α is self-adjoint. (Symmetry)
- (R2) $G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0$ for every $\alpha, \beta > 0$. (Resolvent equation)
- (R3) $\|\alpha G_\alpha f\|_2 \leq \|f\|_2$ for every $\alpha > 0, f \in \ell^2(V, m)$. (Contractivity)
- (R4) $\|\alpha G_\alpha f - f\|_2 \rightarrow 0$ as $\alpha \rightarrow \infty$ for every $f \in \ell^2(V, m)$. (Strong continuity)
- (R5) $0 \leq \alpha G_\alpha f \leq 1$ for $f \in \ell^2(V, m)$ with $0 \leq f \leq 1$. (Markov property)

Every Dirichlet form Q is in one to one correspondence with a non-negative self-adjoint operator L , a strongly continuous Markovian resolvent G_α and a strongly continuous Markovian semigroup T_t . Given any one of those four objects, one can reconstruct the others. We want to give a short discussion about the connection between those objects. For detailed proofs of the below statements see Chapter 1.3/1.4 of [FOT].

Given a Dirichlet form Q , the domain of its associated operator L is given by

$$D(L) = \{u \in D(Q) | \exists w \in \ell^2(V, m) \forall v \in D(Q) : Q(u, v) = \langle w, v \rangle\},$$

on which it acts as

$$Lu = w.$$

This operator is self-adjoint and positive. Furthermore, its square root satisfies $D(L^{1/2}) = D(Q)$ and

$$Q(u, v) = \langle L^{1/2}u, L^{1/2}v \rangle$$

for all $u, v \in D(Q)$. Because L is positive, its spectrum must be contained in $[0, \infty)$. Thus, for positive α the operators $(L + \alpha)^{-1}$ exist and are bounded. They satisfy (R1)-(R5). The spectral calculus of L allows us to define e^{-tL} which is a semigroup satisfying (S1)-(S5). Let us stress some more relations of the above objects.

(i) Let $u \in \ell^2(V, m)$. Then

$$Q(w, v) + \alpha \langle w, v \rangle = \langle u, v \rangle$$

holds for all $v \in D(Q)$ if and only if $w = (L + \alpha)^{-1}u$.

(ii) The domain of L is given by

$$D(L) = \left\{ u \in \ell^2(V, m) \mid \lim_{t \rightarrow 0} \frac{u - e^{-tL}u}{t} \text{ exists} \right\},$$

and for $u \in D(L)$

$$Lu = \lim_{t \rightarrow 0} \frac{u - e^{-tL}u}{t}$$

holds.

Resolvents and semigroups associated with Dirichlet forms may be uniquely extended to bounded operators on $\ell^1(V, m)$ and $\ell^\infty(V)$. Let us discuss this extension for the resolvents. Let $u \in \ell^1(V, m) \cap \ell^2(V, m)$ and $K \subset V$ finite. Then, using the self-adjointness and the Markov property of $(L + \alpha)^{-1}$, we obtain

$$\begin{aligned} \sum_{x \in K} |(L + \alpha)^{-1}u(x)|m(x) &\leq \sum_{x \in V} (L + \alpha)^{-1}|u|(x)1_K(x)m(x) \\ &= \sum_{x \in V} |u|(x)(L + \alpha)^{-1}1_K(x)m(x) \\ &\leq \sum_{x \in V} \alpha^{-1}|u|(x)m(x). \end{aligned}$$

Here 1_K denotes the indicator function of the set K . Since K was arbitrary, we infer $\|(L + \alpha)^{-1}u\|_1 \leq \alpha^{-1}\|u\|_1$. Thus, we can extend $(L + \alpha)^{-1}$ uniquely to $\ell^1(V, m)$. Now let $u \in \ell^\infty(V)$ be positive. Then we can choose a sequence of non-negative functions

$(u_n) \subseteq \ell^2(V, m)$ converging monotonously towards u . Because $(L + \alpha)^{-1}$ maps non-negative functions onto non-negative functions we infer

$$0 \leq (L + \alpha)^{-1}u_n \leq (L + \alpha)^{-1}u_{n+1}.$$

Furthermore, by (R5) we obtain

$$(L + \alpha)^{-1}u_n \leq \alpha^{-1} \|u\|_\infty.$$

Hence the limit as $n \rightarrow \infty$ exists and is bounded by $\alpha^{-1} \|u\|_\infty$. It is easy to verify that this limit is independent of the choice of the sequence u_n . Now set

$$(L + \alpha)^{-1}u(x) := \lim_{n \rightarrow \infty} (L + \alpha)^{-1}u_n(x).$$

For an arbitrary $u \in \ell^\infty(V)$ split u in its positive and negative part and repeat the above procedure. We then obtain a linear operator $(L + \alpha)^{-1} : \ell^\infty(V) \rightarrow \ell^\infty(V)$ satisfying

$$\|\alpha(L + \alpha)^{-1}u\|_\infty \leq \|u\|_\infty.$$

A.2 Dirichlet forms associated with graphs

Theorem A.3. *Let Q be a regular Dirichlet form on $\ell^2(V, m)$. Then, there exists a graph (b, c) over V such that $Q = Q_{b,c}^{(D)}$.*

Proof. Theorem 7 of [KL]. □

The next Theorem is an approximation result for the resolvent $(L^{(D)} + \alpha)^{-1}$. We will need to fix some notation first. For finite $W \subseteq V$ let $L_W = p_W \tilde{L} i_W : C(W) \rightarrow C(W)$. Here i_W is the canonical embedding of $C(W)$ into $C(V)$ and p_W the projection of $C(V)$ onto $C(W)$. In some sense L_W is the restriction of \tilde{L} to $C(W)$, i.e. for $u \in C(W)$ and $x \in W$ we obtain

$$L_W u(x) = \frac{u(x)}{m(x)} \sum_{y \in V} b(x, y) - \frac{1}{m(x)} \sum_{y \in W} b(x, y) u(y) + \frac{c(x)}{m(x)} u(x).$$

Then the following holds.

Theorem A.4. *Let $Q^{(D)}$ be the regular Dirichlet form associated with (b, c) and let $L^{(D)}$ be the associated operator. Let (K_n) be a sequence of finite subsets of V such that $K_n \subseteq K_{n+1}$ and $\cup K_n = V$. Then for any $u \in C(K_1)$*

$$\lim_{n \rightarrow \infty} \|(L^{(D)} + \alpha)^{-1}u - (L_{K_n} + \alpha)^{-1}u\|_2 = 0.$$

(Here u and $(L_{K_n} + \alpha)^{-1}u$ are continued by 0 outside of their domains).

Proof. Proposition 2.7 of [KL]. □

Definition A.5. A bounded operator T on $\ell^2(V, m)$ is called positivity improving if $u \geq 0$ and $u \not\equiv 0$ implies $Tu(x) > 0$ for all $x \in V$.

Theorem A.6. Let (b, c) be connected and Q be a Dirichlet form associated with (b, c) . Then its corresponding resolvent $(L + \alpha)^{-1}$ and its corresponding semigroup e^{-tL} are positivity improving.

Proof. Theorem 6.3 in [HKLW]. □

A.3 Vector valued integration

The following theorems are special cases of statements for Bochner integrals. Since we wanted to avoid Hilbert space valued integration, we include elementary proofs for them. Let us fix some notation. Let $f : V \times [a, b] \rightarrow \mathbb{R}$, such that for each $x \in V$ the function $f(x, \cdot)$ is integrable. We define

$$\int_a^b f(\cdot, t) dt : V \rightarrow \mathbb{R}$$

pointwise via

$$\int_a^b f(\cdot, t) dt(x) := \int_a^b f(x, t) dt.$$

Then the following holds.

Theorem A.7. Assume $a, b \in \mathbb{R}$. Let $f : V \times [a, b] \rightarrow \mathbb{R}$, such that for each $t \in [a, b]$ the function $f(\cdot, t)$ belongs to $\ell^2(V, m)$ and $t \mapsto f(\cdot, t)$ is continuous as a mapping from $[a, b]$ to $\ell^2(V, m)$. Then

$$\left\| \int_a^b f(\cdot, t) dt \right\|_2 \leq \int_a^b \|f(\cdot, t)\|_2 dt.$$

Proof. The continuity assumption ensures that all occurring integrals exist. Without loss of generality we can assume $m \equiv 1$. By monotone convergence it suffices to show the statement for finite sets V . This case can be reduced to $|V| = 2$ by induction. Assume we have shown the result for all sets of cardinality less or equal to n , where $n \geq 2$, and let

$|V| = n + 1$. Fix $o \in V$. Then

$$\begin{aligned}
\left\| \int_a^b f(\cdot, t) dt \right\|_2 &= \left(\sum_{x \in V} \left| \int_a^b f(x, t) dt \right|^2 \right)^{1/2} \\
&= \left(\left| \int_a^b f(o, t) dt \right|^2 + \sum_{x \in V \setminus \{o\}} \left| \int_a^b f(x, t) dt \right|^2 \right)^{1/2} \\
(\text{Ind. hypothesis}) &\leq \left(\left| \int_a^b f(o, t) dt \right|^2 + \left\{ \int_a^b \left[\sum_{x \in V \setminus \{o\}} |f(x, t)|^2 \right] dt \right\}^{1/2} \right)^2^{1/2} \\
(\text{Ind. hypothesis}) &\leq \int_a^b \left(\sum_{x \in V} |f(x, t)|^2 \right)^{1/2} dt \\
&= \int_a^b \|f(\cdot, t)\|_2 dt.
\end{aligned}$$

Now let us treat the case $|V| = 2$ to finish the proof. Our continuity assumptions ensure that all of the above integrals can be computed via Riemann sums. Therefore it suffices to show the statement for simple functions f, g of the form

$$f = \sum_i \alpha_i 1_{A_i}$$

and

$$g = \sum_i \beta_i 1_{A_i}$$

with pairwise disjoint sets A_i . We need to show

$$\left(\left| \int_a^b f(t) dt \right|^2 + \left| \int_a^b g(t) dt \right|^2 \right)^{1/2} \leq \int_a^b (f(t)^2 + g(t)^2)^{1/2} dt.$$

Plugging in f and g and taking the square on both sides of the above inequality, we conclude that it is equivalent to

$$\sum_{i,j} (\alpha_i \alpha_j + \beta_i \beta_j) \lambda(A_i) \lambda(A_j) \leq \sum_{i,j} (\alpha_i^2 + \beta_i^2)^{1/2} (\alpha_j^2 + \beta_j^2)^{1/2} \lambda(A_i) \lambda(A_j),$$

where λ denotes the Lebesgue measure. But this inequality holds since

$$\alpha_i \alpha_j + \beta_i \beta_j \leq (\alpha_i^2 + \beta_i^2)^{1/2} (\alpha_j^2 + \beta_j^2)^{1/2}$$

is always true (use that $(c - d)^2 \geq 0$ for arbitrary $c, d \in \mathbb{R}$). This finishes the proof. \square

Theorem A.8. Assume $a, b \in \mathbb{R}$. Let $f : V \times [a, b] \rightarrow \mathbb{R}$, such that for each $t \in [a, b]$ the function $f(\cdot, t)$ belongs to $\ell^2(V, m)$ and $t \mapsto f(\cdot, t)$ is continuous as a mapping from $[a, b]$ to $\ell^2(V, m)$. Furthermore, let T be a bounded linear operator on $\ell^2(V, m)$. Then $\int_a^b f(\cdot, t)dt \in \ell^2(V, m)$ and

$$T \int_a^b f(\cdot, t)dt = \int_a^b T f(\cdot, t)dt.$$

Proof. $\int_a^b f(\cdot, t)dt \in \ell^2(V, m)$ follows from Theorem A.7 because $t \mapsto \|f(\cdot, t)\|_2$ is continuous and

$$\left\| \int_a^b f(\cdot, t)dt \right\|_2 \leq \int_a^b \|f(\cdot, t)\|_2 dt.$$

Let $g \in \ell^2(V, m)$. Then Lebesgue's theorem yields

$$\int_a^b \langle f(\cdot, t), g \rangle dt = \langle \int_a^b f(\cdot, t)dt, g \rangle.$$

Now the statement follows from

$$\begin{aligned} \langle T \int_a^b f(\cdot, t)dt, g \rangle &= \langle \int_a^b f(\cdot, t)dt, T^* g \rangle \\ &= \int_a^b \langle f(\cdot, t), T^* g \rangle dt \\ &= \int_a^b \langle T f(\cdot, t), g \rangle dt \\ &= \langle \int_a^b T f(\cdot, t)dt, g \rangle, \end{aligned}$$

where T^* denotes the adjoint of T . □

Appendix B

List of Symbols

$a \vee b$	the minimum of a and b	6
$a \wedge b$	the maximum of a and b	6
$\ell^\infty(V)$	bounded functions on V	11
$\ell^p(V, m)$	p -integrable functions on V	11
$\ \cdot\ _Q$	form norm	12
$\ \cdot\ _o$	norm of the Yamasaki space	9
(b, c)	weighted graph over V	5
$C(V)$	all real valued functions on V	6
$C_c(V)$	functions of finite support	7
$\deg(x)$	generalized vertex degree of x	6
δ_x	indicator function of the set $\{x\}$	7
\tilde{D}	functions of finite energy	7
\mathbf{D}	Yamasaki space	9
\tilde{F}	maximal domain of \tilde{L}	7
G	0-th order resolvent operator	17
\tilde{L}	formal operator	7
m	measure of full support on V	11
$Q^{(D)}$	regular Dirichlet form	12
$D(Q)_e$	extended Dirichlet space of Q	21
$Q^{(N)}$	Neumann form	11

\tilde{Q}	maximal quadratic form	6
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